Published by Institute of Physics Publishing for SISSA/ISAS
Received: *May 1, 2002*
Revised: *May 29, 2002*
Accepted: *May 29, 2002*

High energy QCD: stringy picture from hidden integrability

Alexander Gorsky,[ab] **Ian I. Kogan**[acde] **and Gregory Korchemsky**[e]

[a]*Institute of Theoretical and Experimental Physics*
B. Cheremushkinskaya 25, Moscow, 117259, Russia
[b]*LPTHE, Université Paris VI, 4 Place Jussieu, Paris, France*
[c]*Theoretical Physics, Department of Physics, Oxford University*
1 Keble Road, Oxford, OX1 3NP, UK
[d]*IHES, 35 route de Chartres 91440, Bures-sur-Yvette, France*
[e]*Laboratoire de Physique Théorique, Université Paris XI*
91405 Orsay Cedex, France
E-mail: gorsky@heron.itep.ru, i.kogan1@physics.ox.ac.uk, korchems@th.u-psud.fr

Abstract: We discuss the stringy properties of high-energy QCD using its hidden integrability in the Regge limit and on the light-cone. It is shown that multi-colour QCD in the Regge limit belongs to the same universality class as superconformal $\mathcal{N}=2$ SUSY YM with $N_f = 2N_c$ at the strong coupling orbifold point. The analogy with integrable structure governing the low energy sector of $\mathcal{N}=2$ SUSY gauge theories is used to develop the brane picture for the Regge limit. In this picture the scattering process is described by a single membrane wrapped around the spectral curve of the integrable spin chain and unifying hadrons and reggeized gluons involved in the process. New quasiclassical quantization conditions for the complex higher integrals of motion are suggested which are consistent with the S-duality of the multi-reggeon spectrum. The derivation of the anomalous dimensions of the lowest twist operators is formulated in terms of the Riemann surfaces.

Keywords: Duality in Gauge Field Theories, Brane Dynamics in Gauge Theories, Lattice Integrable Models, QCD.

Contents

1. Introduction

During the recent years it was recognized that there is the hidden integrability behind the evolution equations in asymptotically free Yang-Mills theories [1, 2]. It was found that high-energy asymptotics of scattering amplitudes in QCD in the Regge limit are described by evolution equations which are ultimately related to the $\mathrm{SL}(2,\mathbb{C})$ XXX Heisenberg integrable spin chains. More recently, a similar connection was observed for another physically interesting limit, namely the QCD dynamics on the light-cone. There, the logarithmic evolution of the composite light-cone operators is connected with the real $\mathrm{SL}(2,\mathbb{R})$ XXX Heisenberg spin chain and the spectrum of anomalous dimensions of those operators was identified with the spectrum of this magnet [3, 4, 5, 6]. For the time being it seems that the phenomena is quite general and the effective QCD dynamics in different kinematical limits is integrable indeed.

On the other hand, there is another class of (supersymmetric) Yang-Mills theories in which integrability emerges. In the $\mathcal{N}=2$ SUSY YM theory which can be solved in the low-energy limit exactly [7] the relevant integrable system was identified as the complexified periodic Toda chain. The low-energy effective action and the BPS spectrum in the theory can be described in terms of the classical Toda system [8]. When one adds the fundamental matter with $N_f < 2N_c$, which does not change the asymptotically free nature of the theory, the universality class of integrable model is changing. Instead of the classical Toda system one gets the classical XXX spin chain [9]. In the case of the superconformal $N_f = 2N_c$ theory, the solution leads to the periodic XXX chain in the magnetic field [10]. Because these magnets are classical, the spins are not quantized and their values are determined by the matter masses. In the special case, when the matter is massless, one arrives at the classical XXX magnet of spin zero.

At the first glance there is nothing in common neither between high-energy (Regge) limit of QCD and low-energy limit of SUSY YM, nor between the corresponding integrable models. In this paper we shall demonstrate the opposite and show that there is a deep relation between the Regge limit of QCD and some particular limit of superconformal $\mathcal{N} = 2$ SUSY YM at $N_f = 2N_c$. This relation is based on the fact that in the both cases we are dealing with the same class of integrable models — the $\mathrm{SL}(2,\mathbb{C})$ homogenous spin zero Heisenberg magnets. However, there is the key difference between two theories. The integrable model in the SUSY YM is a classical Heisenberg magnet, whereas the Regge limit of QCD is described by a quantum Heisenberg magnet. To bring two pictures together, it is natural to develop quasiclassical description in QCD case and quantize the classical picture in SUSY YM. The first issue was considered in [11, 12, 13] while some arguments concerning the quantization of the integrable system related to SUSY YM theories can be found in [14].

The central role in our approach will be played by Riemann surfaces which appear naturally as spectral curves in a description of classical spin chains. To some extent the Riemann surface fixes the universality class of the classical integrable model and it defines general solution to the classical equations of motion. This solution is a finite-gap soliton expressed in terms of θ-functions on the Riemann surface. This surface has a natural complex structure whose moduli are parametrized by the integrals of motion. The classical dynamics is linearized on the jacobian of this Riemann surface. In the case of the SUSY YM these Riemann surfaces encode the expression for the low energy effective actions [9, 10]. The moduli of the complex structure are fixed by the vacuum expectation values of the scalar in $\mathcal{N} = 2$ theory. One gets Riemann surfaces in the quasiclassical limit of the Schrödinger equation for the quantum spin chain related to QCD in the Regge limit [12, 13]. The main goal of this paper is to use these Riemann surfaces to develop the stringy representation for the effective dynamics of QCD in the Regge limit exploring the known stringy realization of low-energy regime of SUSY YM.

Our starting point is the observation that the Seiberg-Witten curve for the superconformal $\mathcal{N} = 2$ SUSY YM theory with gauge group $\mathrm{SU}(N_c)$ and the number of flavours $N_f = 2N_c$ coincides with the spectral curve for Heisenberg spin magnet describing multi-colour QCD in the Regge limit. On the SUSY YM side, we still have several unidentified parameters. The first one is the rank of the gauge group $\mathrm{SU}(N_c)$. On the QCD side, in the multi-colour limit, we also have one additional parameter, namely the number of reggeized gluons $N = 2, 3, \ldots$ exchanged in the t-channel [15, 16, 17]. It will be shown later that the correspondence between these two theories implies that the number of reggeized gluons $N = N_c$. However the superconformal theory has additional freedom due to the arbitrary value of the bare coupling constant $\tau_0 = i/g^2 + 2\pi\theta$ and the Seiberg-Witten curves are bundled over the complex τ_0 plane. On the complex τ_0-plane there are singular points at which the discriminant of the curve vanishes, i.e. the spectral curve becomes degenerate. It turns out that the Regge limit of the multi-colour QCD is related to one of these singular points, the so-called strong coupling orbifold point [18, 19] which is S-dual to another singular point corresponding to the weak coupling $e^{2\pi i\tau_0} \to 0$ regime. Let us emphasize that in the superconformal theory the effective coupling constant does not coincide with the bare one, because there are finite perturbative one-loop and nonperturbative instanton corrections to the low-energy effective action. At the strong coupling regime under consideration one has to take into account contributions from all numbers of instantons.

After identification of the universality class one can investigate relevant S-duality properties of both theories, which can be read off from the spectral curve. In the SUSY case, the S-duality is the famous electric-magnetic duality. The Seiberg-Witten solution follows from the compatibility of the duality with the renormalization group (RG) flow of the effective action on the moduli space of the theory. In the QCD case the situation is more subtle since contrary to the SUSY case we have to deal with the quantum dynamics of the same integrable system and one has to formulate the precise meaning of the duality at the quantum level. It turns out that the duality manifests itself in the spectrum of the integrals of motion [13]. We formulate a new WKB quantization conditions obeying the duality property and demonstrate that their solutions are in a good agreement with the numerical calculations of the spectrum of the three reggeon system [20]. This strongly support the conjecture that the notion of duality survives at the quantum level. In result we predict that the quantum spectrum of the N-reggeon system in the multi-colour QCD is parametrized by two $(N-2)-$dimensional vectors $\vec{n}$ and $\vec{m}$, which have the meaning of "electric" and "magnetic" quantum numbers.

We reformulate the above mentioned WKB quantization conditions in stringy terms using the brane realization of the BPS spectrum in superconformal $\mathcal{N} = 2$ SUSY YM theory [21, 22]. Formally it is a simple task to identify the corresponding geometrical objects however the proper meaning of the "magnetic" quantum number in the Regge limit is a subtle point. The duality combined with RG behaviour give rise to brane realization of the ground state in the SUSY YM theories. There are two apparently different IIA/M and IIB/F pictures. In the IIA/M picture the ground state is described by the M5-brane with the world-volume $R^4 \times \Sigma$ where Σ is the spectral curve of the corresponding integrable system [21, 23]. The stable BPS states correspond to the M2-branes ending on this M5-brane. In the IIB/F picture [22] one has to consider a D3-brain in the orientifold background and BPS states are described by the dyonic strings stretched between the D3-brane and orientifold.

In this paper we shall start to develop a similar brane picture for the Regge limit of the multi-colour QCD. We shall argue that the dynamics of the multi-reggeon compound states in multi-colour QCD is described in this picture by a single membrane wrapped around the spectral curve of the Heisenberg magnet. The fact that we are dealing with the *quantum* magnet implies that the moduli of the complex structure of the spectral curve can take only quantized values. We shall present a raw of arguments supporting this picture. Let us note that in the Regge limit there is a natural splitting of the four-momenta of reggeized gluons into two transverse and two longitudinal components, which is unambiguously determined by the kinematics of the scattering process. One can make a Fourier transform with respect to the transverse momenta and define a mixed representation: two-dimensional longitudinal momentum space and two-dimensional impact parameter space for which it is natural to use complex coordinates z and $\bar{z} = z^*$. The Riemann surface which the brane is wrapped around lives in this "mixed" coordinate-momenta space. Different projections of the wrapped brane represent hadrons and reggeons. The genus of this surface is fixed by the number of reggeons. Let us note that multi-reggeon states appear from the summation of only planar diagrams in QCD and from the point of view of topological expansion they all correspond to the cylinder-like diagram in a colour space. It contribution to the scattering amplitude is given by the sum of N-reggeon exchanges in the t-channel with $N = 2, 3 \ldots$. It is amusing that in result we get a picture where we sum over Riemann surfaces of arbitrary genus $g = N - 2$.

Finally, we make some comments on a dual (super)gravity realization of the Regge limit of multi-colour QCD. Our approach is different from the recent studies [24, 25, 26] of high-energy scattering in QCD based on the AdS_5 geometry within the AdS/CFT correspondence [27]. We speculate that a reggeized gluon in QCD can be identified as a singleton in the AdS_3 space.

The paper is organized as follows. In section 2 we remind how the integrable spin chains appear in the Regge limit of QCD. In section 3 we review the main features of the finite-gap solutions to the classical equation of the motion in the periodic spin chains which are relevant for our consideration. We also discuss their quantization using the method of Separated Variables [28]. In section 4 we identify the universality class of the Regge limit of multi-colour QCD by comparing the Riemann surfaces in QCD and $\mathcal{N} = 2$ SUSY YM. Based on this identification, we make a conjecture about brane realization of QCD in the Regge limit. In section 5 we investigate the duality properties of the multi-reggeon states and provide WKB like description of their spectrum in terms of Riemann surfaces. In section 6 we propose the stringy/brane picture for the description of the anomalous dimensions of the lowest twist operators in QCD. In section 7 several comments concerning the supergravity dual description of the Regge limit are presented. The obtained results are discussed in the conclusion.

2. Regge asymptotics in QCD and Heisenberg magnets

It is well known from the old days of the Regge theory that the scattering amplitudes of hadrons grow at high-energy. Useful framework for understanding this phenomenon in QCD comes from the studies of the asymptotic behaviour of the scattering amplitude of two hadronic states consisting of a pair of heavy quarks [29], the so-called onia system. The advantage of such system is that it captures all essential features of the Regge phenomenon in QCD and, at the same time, the scattering amplitudes can be calculated perturbatively order-by-order in the QCD coupling expansion. Denoting the invariant mass of the scattering onia by Q^2 and their center-of-mass energy by s, we introduce the dimensionless Bjorken scaling variable $x = Q^2/s$. The Regge limit corresponds to the asymptotics at large $s = Q^2/x$ with fixed Q^2, or equivalently to small x.

One expects that the calculation of the high-energy asymptotics of the onium-onium cross-section should eventually match the Regge model prediction. The Regge model interprets the growth of the scattering amplitudes at high energy s, or equivalently at small x, by introducing the notion of the Pomeron as the Regge pole with the quantum numbers of the QCD vacuum. Its contribution of the onium-onium cross-section at small x takes the form

$$\sigma_{\rm tot}(x,Q^2) = x^{-(\alpha_{\mathbb{P}}-1)}\,\beta_A^{\mathbb{P}}(Q^2)\,\beta_B^{\mathbb{P}}(Q^2)\,, \tag{2.1}$$

with $\alpha_{\mathbb{P}}$ the Pomeron intercept and $\beta^{\mathbb{P}}_{A,B}(Q^2)$ the residue factors corresponding to the onia A and B, respectively. Despite the fact that the Regge model provides a successful phenomenological description of the experimental data in terms of "hard" (perturbative) and "soft" (nonperturbative) Pomerons its status within QCD remains unclear.

The first attempts to understand the "hard" Pomeron within perturbative QCD have been undertaken more than 20 years ago and they have led to the discovery of the BFKL Pomeron [30]. In the BFKL approach, two onia scatter each other by exchanging soft gluons in the t-channel. Calculating the corresponding Feynman diagrams in powers of $\alpha_{\rm s}$ one obtains the following general form of the perturbative expansion of the cross-section at small x and fixed Q^2, [15, 16, 17]

$$\sigma_{\rm tot}(x,Q^2) = \sigma_{\rm Born}\sum_{m=0}^{\infty}\sum_{n=0}^{m}(\alpha_{\rm s}N_c)^{m-n}(\alpha_{\rm s}N_c\ln x)^n f_{m,n}(Q^2)\,, \tag{2.2}$$

with $\sigma_{\rm Born}$ being the cross-section at the Born level. Here, the double sum involves two parameters of the expansion — the QCD coupling constant, $\alpha_{\rm s} \ll 1$, and the energy dependent parameter $\alpha_{\rm s}\ln 1/x$ that becomes large at small x. The coefficient functions $f_{m,n}(Q^2)$ depend on the invariant mass of the onia, Q^2, as well as on the number of colours, $1/N_c^2$. They can be calculated perturbatively for large invariant mass $Q^2 \gg \Lambda^2_{\rm QCD}$ by making use of the wave functions of the onia states. The number of different terms in (2.2) rapidly increases to higher orders in $\alpha_{\rm s}$ and in order to find $\sigma_{\rm tot}(x,Q^2)$ at small x one has to develop a meaningful approximation which, firstly, correctly describes the small$-x$ asymptotics of the infinite series (2.2) and, secondly, preserves the unitarity constraint

$$\sigma_{\rm tot}(x,Q^2) < {\rm const.}\times\ln^2\left(\frac{1}{x}\right) \tag{2.3}$$

as $x\to 0$. To satisfy the first condition, one can neglect in (2.2) the terms containing $f_{m,n}$ with $n<m$ as suppressed by powers of $\alpha_{\rm s}$ with respect to the leading term $f_{m,m}$. The resulting series defines the onium-onium cross-section in the leading logarithmic approximation (LLA). It can be resummed to all orders in $\alpha_{\rm s}$ leading to the BFKL pomeron [30]

$$\sigma^{\rm LLA}_{\rm tot}(x,Q^2) = \sigma_{\rm Born}\sum_{m=0}^{\infty}(\alpha_{\rm s}N_c\ln x)^m f_{m,m}(Q^2) \sim \frac{x^{-\alpha_{\rm s}N_c 4\ln 2/\pi}}{\sqrt{\alpha_{\rm s}N_c\ln 1/x}}\,. \tag{2.4}$$

The BFKL Pomeron leads to unrestricted rise of the cross-section and violates the unitary bound (2.3). In order to preserve the unitarity of the S-matrix of QCD and fulfill (2.3), one has to take into account an *infinite* number of nonleading terms in (2.2). This means that with the unitarity condition taken into account the series (2.2) does not have small parameter of expansion like α_s. Instead of searching for this parameter one may start with the LLA result (2.4) and try to identify the nonleading terms in (2.2), which should be added to (2.4) in order to restore the unitarity. In this way, one arrives at the generalized leading logarithmic approximation [17, 15, 16]. One should keep in mind, however, that in this approximation the unitarity is preserved only in the direct channels of the process but not in subchannels corresponding to the different groups of particles in the final state.

2.1 Generalized LLA

In the generalized LLA, one uses the remarkable property of gluon reggeization [30] in order to formulate a new diagram technique for calculation of the onium-onium scattering amplitudes [17, 15, 16]. Namely, an infinite set of standard Feynman diagrams involving "bare" gluons can be replaced by a few reggeon diagrams describing propagation of reggeized gluons, or reggeons, and their interaction with each other. Each reggeon diagram appears as a result of resummation of an infinite number of Feynman diagrams with "bare" gluons. In this way, one defines an effective theory in which the reggeized gluons start play a role of a new elementary fields. In this theory, the scattering amplitudes describe the effects of the propagation of the reggeized gluons in the t-channel and their interaction with each other. As usual, each diagram in the effective theory is equivalent to an infinite sum of the original QCD Feynman diagrams.

The resulting set of the reggeon diagrams defining the onium-onium cross-section in the generalized LLA is shown in figure 1. According to the optical theorem, the cross-section is given by an imaginary part of their contribution. The vertical curly lines represent the reggeons propagating in the t-channel and the triple gluon vertices describe the effective BFKL interaction between the reggeons and the gluons propagating in the $s-$channel. Upper and lower blobs denote the coupling of the reggeons to two onia states A and B.

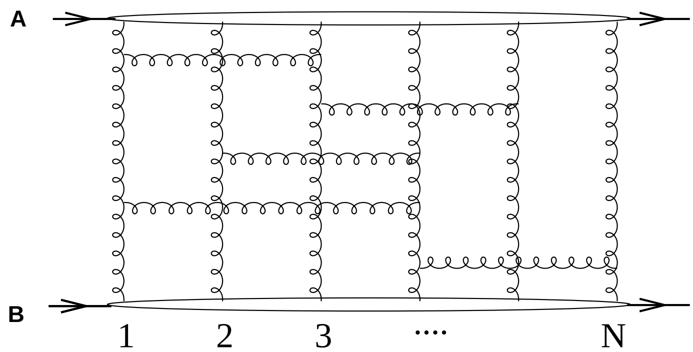

Figure 1: The Feynman diagrams contributing to the scattering amplitude of the onium-onium scattering in the generalized LLA and describing the N reggeon exchange in the t-channel.

One of the peculiar features of the reggeon scattering in the generalized LLA is that it is elastic and pair-wise. The number of reggeons exchanged in the t-channel is not changed. This allows us to interpret the diagrams shown in figure 1 as describing the propagation of the conserved number $N = 2, 3, \dots$ of pair-wisely interacting reggeons. At $N = 2$ the diagram in figure 1 determines the leading logarithmic result for the scattering amplitude, eq. (2.4), and it contributes to the coefficient functions $f_{m,m}(Q^2)$ in (2.4) to all orders of perturbation theory. The contribution of the diagram with $N = 3$ is suppressed by a power of $\alpha_{\rm s}$ with respect to that for $N = 2$ and it determines the first nonleading coefficient $f_{m,m-1}$ in (2.2).

The calculation of the N-reggeon diagrams can be performed using the Bartels-Kwieciński-Praszałowicz approach [15, 16]. The diagrams shown in figure 1 have a generalized ladder form and for fixed number of reggeons, N, their contribution satisfies Bethe-Salpeter like equations for the scattering of N particles. The solutions to these equations define the colour-singlet compound states $|\Psi_N\rangle$ built from N reggeons. These states propagate in the t-channel between two scattered onia states and give rise to the following (Regge like) high energy asymptotics of the the onium-onium cross-section

$$\sigma_{\rm tot}(s) \sim \sum_{N=2,3,\dots} (\alpha_{\rm s} N_c)^N \frac{x^{-\alpha_{\rm s} N_c \varepsilon_N/\pi}}{\sqrt{\alpha_{\rm s} N_c \ln 1/x}} \beta_A^{(N)}(Q^2) \beta_B^{(N)}(Q^2) \tag{2.5}$$

with the exponents ε_N defined below. Here, each term in the sum is associated with the contribution of the N-reggeon compound states. At $N = 2$, the corresponding state defines the BFKL pomeron (2.4) with $\varepsilon_2 = 4 \ln 2$.

The N-reggeon states are defined in QCD as solutions to the Schrödinger equation

$$\mathcal{H}_N \Psi(\vec{z}_1, \vec{z}_2, \dots, \vec{z}_N) = \frac{\alpha_{\rm s}}{\pi} N_c \varepsilon_N \Psi(\vec{z}_1, \vec{z}_2, \dots, \vec{z}_N) \tag{2.6}$$

with the effective QCD hamiltonian $\mathcal{H}_N$ acting on two-dimensional transverse coordinates of reggeons, $\vec{z}_k$ $(k = 1, \dots, N)$ and their colour SU(N_c) charges

$$\mathcal{H}_N = -\frac{\alpha_{\rm s}}{2\pi} \sum_{1 \le i < j \le N} H_{ij}\, t_i^a t_j^a \,. \tag{2.7}$$

Here, the sum goes over all pairs of reggeons. Each term in this sum is factorized into the product of two operators acting on the color indices and the two-dimensional coordinates. The former is given by the direct product of the gauge group generators in the adjoint representation of the SU(N_c) group, acting in the color space of $i-$th and $j-$th reggeons,

$$t_i^a = \underbrace{I \otimes \cdots \otimes t^a}_{i} \otimes \cdots \otimes I \,, \qquad (t^a)_{bc} = -i f_{abc} \,,$$

with f_{abc} the structure constants of the SU(N_c). The operator H_{ij} acts only on the transverse coordinates of reggeons, $\vec{z}_i$ and $\vec{z}_j$. The hamiltonian $\mathcal{H}_N$ commutes with the total color charge of the system of N-reggeons. Solving (2.6), we are looking for the colour-singlet states, that is the eigenstates with the total color charge equal to zero, $\sum_{i=1}^N t_i^a |\Psi_N\rangle = 0$. The residue factors in (2.5) measure the overlap of $\Psi(\vec{z}_1, \vec{z}_2, \dots, \vec{z}_N)$ with the wave functions of two scattered particles, $\beta_A^{(N)}(Q^2) = \langle A|\Psi_N\rangle$ and similar for $\beta_B^{(N)}$.

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To get some insight into the properties of the N-reggeon states, it proves convenient to interpret the Feynman diagrams shown in figure 1 as describing a quantum-mechanical evolution of the system of N particles in the t-channel between two onia states $|A\rangle$ and $|B\rangle$

$$\sigma_{\rm tot}(s) = \sum_{N\geq 2} (\alpha_{\rm s} N_c)^N \langle A| \, {\rm e}^{\ln(1/x)\cdot \mathcal{H}_N} \, |B\rangle \,, \tag{2.8}$$

with the rapidity $\ln x = \ln Q^2/s$ serving as euclidean evolution time. To find the high energy asymptotics of (2.8), one has to solve the Schrödinger equation (2.6) for arbitrary number of reggeized gluons N, expand the operator $(1/x)^{\mathcal{H}_N}$ over the complete set of the eigenstates of $\mathcal{H}_N$ and, then, resum their contribution for arbitrary N. In this way, one arrives at (2.5).

It follows from (2.8) that for fixed number of the reggeons, N, and large evolution time, $\ln 1/x \gg 1$, the dominant contribution to $\sigma_{\rm tot}(s)$ comes from the eigenstates of $\mathcal{H}_N$ with the maximal energy. This allows us to identify the exponent $(-\varepsilon_N)$ in (2.5) as the energy of the ground state of the hamiltonian $(-\mathcal{H}_N)$. It turns out that the energy spectrum of the reggeon hamiltonian is gapless and the contribution of the eigenstates next to the ground state amounts to the appearance of the additional factor $\sim (\alpha_{\rm s} \ln 1/x)^{-1/2}$ in the r.h.s. of (2.5). The contribution of the N-reggeon state to $\sigma_{\rm tot}(s)$, eq. (2.5), depends crucially on the sign of the energy — for $\varepsilon_N > 0$ (or $\varepsilon_N < 0$) it increases (or decreases) as $s \to \infty$.

Finding the spectrum of the compound states (2.6) for arbitrary number of reggeized gluons N and eventual resummation of their contribution to the scattering amplitude (2.5) is the outstanding theoretical problem in high-energy QCD. At $N = 2$, the solution to (2.6) has been found a long time ago — the well-known BFKL Pomeron [30], $\varepsilon_2 = 4\ln 2$. At $N = 3$ the solution to (2.6) — the Odderon state in QCD [31], was formulated only a few years ago [32], $\varepsilon_3 = -0.2472\ldots$, by making use of the remarkable properties of integrability of the effective QCD hamiltonian [1, 2]. Recently, a significant progress has been achieved in solving (2.6) for arbitrary number of reggeons in the milti-colour limit [33, 34, 20].

2.2 Large-N_c limit

At large N_c one can expand the matrix elements in the r.h.s. of (2.8) in powers of $1/N_c^2$ and obtain a topological expansion of $\sigma_{\rm tot}(s)$ [35]. The leading term of this expansion corresponds to the planar cylinder diagram in which the top and the bottom disks correspond to two onia states and the walls are formed by N reggeons. In this diagram, in order to preserve the planarity, each reggeon is allowed to interact only with two neighbouring reggeons. Going back from the planar diagrams to the hamiltonian $\mathcal{H}_N$, this leads to simplification of the reggeon interaction in the multi-colour limit. Namely, for $\alpha_{\rm s} N_c =$ fixed and $N_c \to \infty$, the hamiltonian (2.7) can be written as

$$\mathcal{H}_N = \frac{\alpha_{\rm s} N_c}{4\pi} \sum_{1\leq i\leq N} H_{i,i+1} + \mathcal{O}(N_c^{-2}) \tag{2.9}$$

with $H_{N,N+1} \equiv H_{N,1}$. Thus, the pair-wise interaction between reggeons in (2.7) is reduced in the multi-colour limit to a nearest-neighbour interaction (2.9) and, in addition, the colour operator is replaced by a $c-$valued factor $t_i^a t_j^a \to -N_c/2$. As was already mentioned, the two-particle reggeon hamiltonian $H_{i,i+1}$ acts on the two-dimensional (transverse) reggeon coordinates $\vec{z} = (x,y)$ and it coincides with the BFKL kernel [30]. It becomes convenient to introduce the complex valued (anti)holomorphic coordinates, $z = x+iy$ and $\bar{z} = x-iy$, and parameterize the position of the kth reggeon as $\vec{z}_k = (z_k, \bar{z}_k)$. One of the remarkable features of the BFKL kernel is that it is given by the sum of two mutually commuting operators acting separately on $z-$ and $\bar{z}-$coordinates of the reggeons. Making use of this property, one can rewrite the N-reggeon hamiltonian (2.9) as [30]

$$\mathcal{H}_N = \frac{\alpha_{\rm s} N_c}{4\pi}\big(H_N + \bar{H}_N\big) + \mathcal{O}(N_c^{-2})\,. \tag{2.10}$$

Here the hamiltonians H_N and $\bar{H}_N$ act on the (anti)holomorphic coordinates and describe the nearest-neighbour interaction between N reggeons

$$H_N = \sum_{m=1}^{N} H(z_m, z_{m+1})\,, \qquad \bar{H}_N = \sum_{m=1}^{N} H(\bar{z}_m, \bar{z}_{m+1})\,,$$

with the periodic boundary conditions $z_{k+1} = z_1$ and $\bar{z}_{k+1} = \bar{z}_1$. The interaction hamiltonian between two reggeons with the coordinates $(z_1, \bar{z}_1)$ and $(z_2, \bar{z}_2)$ in the impact parameter space, is given by the BFKL kernel

$$H(z_1, z_2) = -\psi(J_{12}) - \psi(1 - J_{12}) + 2\psi(1)\,, \tag{2.11}$$

where $\psi(x) = d\ln\Gamma(x)/dx$ and the operator J_{12} is defined as a solution to the equation

$$J_{12}(J_{12} - 1) = -(z_1 - z_2)^2 \partial_1 \partial_2 \tag{2.12}$$

with $\partial_m = \partial/\partial z_m$. The expression for $H(\bar{z}_1, \bar{z}_2)$ is obtained from (2.11) by substituting $z_k \to \bar{z}_k$.

Since the hamiltonian H_N and $\bar{H}_N$ act along two different directions on the two-dimensional plane, they commute with each other, $[H_N, \bar{H}_N] = 0$. This allows us to reduce the original Schrödinger equation (2.6) to the system of two one-dimensional Schrodinger equations for the hamiltonians H_N and $\bar{H}_N$. For fixed number of reggeons, N, the operator H_N describes the system of N particles on a complex $z-$line interacting with their neighbours through the two-particle hamiltonian (2.11). This quantum-mechanical system can be solved exactly for the system with $N = 2$ particles leading to the BFKL Pomeron. It is not obvious, however, whether the exact solution can be found for an arbitrary number of reggeons. For this to occur, the system of N reggeons should possess an additional symmetry. It turns out that such a hidden symmetry indeed exists [1, 2]. Namely, the Schrödinger equation for the N-reggeon system in multi-colour limit contains the set of the mutually commuting integrals of motion q_k and $\bar{q}_k$ ($k = 2, \ldots, N$) defined as

$$q_k = i^k \sum_{1 \le j_1 < j_2 < \cdots < j_k \le N} z_{j_1 j_2} \cdots z_{j_{k-1}, j_k} z_{j_k, j_1} \partial_{z_{j_1}} \cdots \partial_{z_{j_{k-1}}} \partial_{z_{j_k}} \tag{2.13}$$

with $z_{jk} \equiv z_j - z_k$. The charges $\bar{q}_k$ are given by similar expressions in the $\bar{z}-$sector. The eigenstates $\Psi(\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_N)$ have to diagonalize these operators and the corresponding eigenvalues $q \equiv \{q_k, \bar{q}_k = q_k^*\}$, with $k = 2, \ldots, N$, form the complete set of quantum numbers parameterizing the spectrum of the Schrödinger equation (2.6), $\varepsilon_N = \varepsilon_N(q)$. This implies that the Schrödinger equation (2.6) for the N-reggeon states is completely integrable in the multi-colour limit and it can be solved by applying the powerful Quantum Inverse Scattering Method well-known in the theory of integrable models.

The famous example of the exactly solvable many-body quantum mechanical system is the one-dimensional XXX Heisenberg chain of N interacting spins described by the hamiltonian

$$H_N^{s=1/2} = -\sum_{m=1}^{N} \vec{S}_m \cdot \vec{S}_{m+1}\,,$$

where $\vec{S}_m$ are the SU(2) generators of the spin $s = 1/2$ acting in the $m-$th site and $\vec{S}_{N+1} \equiv \vec{S}_1$. It turns out [2, 33] that this simple model admits nontrivial generalizations for an arbitrary complex value of the spin s. In the latter case, the spin operators $\vec{S}_m$ are the generators of an infinite-dimensional representation of the $\mathrm{SL}(2,\mathbb{C})$ group of the principal series. Remarkably enough, in the special case of the $\mathrm{SL}(2,\mathbb{C})$ spin $s = 0$, the hamiltonian of such completely integrable noncompact XXX Heisenberg magnet becomes identical to the QCD hamiltonian (2.10), describing the interaction between N-reggeons in the multi-color limit.

To explain this correspondence let us consider the one-dimensional lattice with periodic boundary conditions and with the number of sites, N, equal to the number of reggeons. Each site is parameterized by the two-dimensional vector $\vec{z}_m = (z_m, \bar{z}_m)$ with z_m and $\bar{z}_m$ being holomorphic and antiholomorphic coordinates, respectively, and $m = 1, \ldots, N$. In addition, one assign to each site the spins operators $S_k^{\pm,0}$ and $\bar{S}_k^{\pm,0}$, which are the six generators of the principal series of the $\mathrm{SL}(2,\mathbb{C})$ group. They are realized on the quantum space of the model as the differential operators

$$S_k^0 = z_k \partial_{z_k} + s\,, \qquad S_k^- = -\partial_{z_k}\,, \qquad S_k^+ = z_k^2 \partial_{z_k} + 2s z_k\,. \tag{2.14}$$

The operators $\bar{S}_k^{\pm,0}$ are given by similar expressions with z_k, s replaced by $\bar{z}_k$, $\bar{s} = 1 - s^*$, respectively. The pair of complex parameters $(s, \bar{s})$ specifies the $\mathrm{SL}(2,\mathbb{C})$ representation. For the principal series of the $\mathrm{SL}(2,\mathbb{C})$ they are parameterized by integer n_s and real ν_s as $s = (1 + n_s)/2 + i\nu_s$ and $\bar{s} = 1 - s^*$. To match the reggeon hamiltonian, eq. (2.6), the spins have to be equal to $s = 0$ and $\bar{s} = 1$. In this representation, the BFKL kernel defining the interaction between two reggeons describes the interaction between the nearest spins. The hamiltonian of exactly solvable noncompact $\mathrm{SL}(2,\mathbb{C})$ Heisenberg magnet of spin $(s, \bar{s})$ is defined as [2, 33]

$$H_N^{(s)} = \sum_{m=1}^{N} H_{m,m+1}\,, \qquad H_{m,m+1} = -i \frac{d}{du} \ln R_{m,m+1}(u,u)\bigg|_{u=0} \tag{2.15}$$

where the operator $R_{m,m+1}$, the so-called fundamental $R-$matrix for the $\mathrm{SL}(2,\mathbb{C})$ group, satisfies the Yang-Baxter equation and it is given by [33]

$$R_{12}(u,\bar{u}) = \frac{\Gamma(i\bar{u})\Gamma(1+i\bar{u})}{\Gamma(-iu)\Gamma(1-iu)} \times \frac{\Gamma(1-\bar{J}_{12}-i\bar{u})\,\Gamma(\bar{J}_{12}-i\bar{u})}{\Gamma(1-J_{12}+iu)\,\Gamma(J_{12}+iu)}\,, \tag{2.16}$$

with J_{12} defined as the sum of two spins

$$J_{k,k+1}(J_{k,k+1}-1) = (S^{(k)} + S^{(k+1)})^2 \tag{2.17}$$

with $S_\alpha^{(N+1)} = S_\alpha^{(1)}$, and $\bar{J}_{k,k+1}$ is defined similarly. The operator $R_{12}(u,\bar{u})$ acts on the tensor product $V \otimes V$ with $V \equiv V^{(s,\bar{s})}$ being the representation space of the principal series of the $\mathrm{SL}(2,\mathbb{C})$.

Substituting (2.16) into (2.15), we verify that the hamiltonian of noncompact Heisenberg magnet of the $\mathrm{SL}(2,\mathbb{C})$ spin $s = 0$ and $\bar{s} = 1$ coincides with the hamiltonian of reggeon interaction (2.11). This leads to the following identity between two hamiltonians

$$\mathcal{H}_N \equiv \frac{\alpha_{\rm s} N_c}{4\pi} H_N^{(s=0)} + \mathcal{O}\Big(\frac{1}{N_c^2}\Big)\,. \tag{2.18}$$

Thus, the problem of finding the spectum of the N-reggeon states in multi-colour QCD is reduced to solving the Schrödinger equation for a peculiar completely integrable model — the noncompact Heisenberg magnet with spin operators being the generators of the principal series of the $\mathrm{SL}(2,\mathbb{C})$ group of the spin $s = 0$. This model has a number of remarkable properties [2, 33], which make it completely different from conventional Heisenberg spin magnets studied so far. Firstly, the $\mathrm{SL}(2,\mathbb{C})$ representation of the principal series does not have the highest weight and, as a consequence, the Algebraic Bethe Ansatz is not applicable to construction the eigenstates of the model. Secondly, despite the fact that the hamiltonian (2.10) can be split into the sum of holomorphic and antiholomorphic mutually hamiltonians acting on the $z-$ and $\bar{z}-$coordinates of the particles, the two sectors are not independent. The interaction between two sectors occurs through the condition for the wave function to be a single valued function on the $\vec{z} = (z,\bar{z})-$plane. This requirement plays a crucial role in establishing the quantization conditions on the integrals of motion and in finding the energy spectrum of the model [20].

3. Finite gap solutions and their quantization

To get some insight into the properties of the Schrödinger equation (2.6), it becomes convenient to consider a classical analog of the noncompact Heisenberg spin magnet. As was shown in [11, 12], the corresponding classical hamiltonian model naturally appears in the WKB solutions of the Schrödinger equation (2.6). From point of view of classical dynamics, the system describes two copies of one-dimensional spin chains "living" on the complex $z-$ and $\bar{z}-$lines. Each of these spin chains can be analyzed separately and, for simplicity, we shall concentrate in this section on the holomorphic system.

3.1 Classical dynamics

Following the Quantum Inverse Scattering Method, the classical homogenous spin $s = 0$ chain of the length N is defined in the holomorphic sector by the 2×2 Lax matrices

$$L_k(x) = x \cdot \mathbf{1} + \sum_{a=1}^{3} S_k^a \cdot \sigma^a = \begin{pmatrix} x - iz_k p_k & ip_k \\ -iz_k^2 p_k & x + iz_k p_k \end{pmatrix} \tag{3.1}$$

where σ^a are the Pauli matrices and x is the spectral parameter. Here, S_i^a stand for classical spins. They are obtained from the spin operators in the holomorphic sector, eq. (2.14), by substituting the derivatives with respect to $z-$coordinates by the corresponding classical momenta $i\partial_{z_k} \to p_k$

$$q_k\Big|_{\text{classical}} = \sum_{1\le j_1<j_2<\dots<j_k\le N} z_{j_1j_2}\cdots z_{j_{k-1},j_k} z_{j_k,j_1} p_{j_1}\cdots p_{j_{k-1}} p_{j_k} \tag{3.2}$$

In this way, one can define on the phase space of the model the Poisson brackets

$$\{z_k, p_j\} = \delta_{jk} \tag{3.3}$$

and verify that the Poisson brackets of the dynamical variables S_a, $a = 1, 2, 3$ (taking values in the algebra of functions) look like

$$\{S_k^a, S_j^b\} = -i\epsilon_{abc} S_k^c\, \delta_{jk}\,, \tag{3.4}$$

so that $\{S_a\}$ plays the role of angular momentum ("classical spin") giving the name "spin-chains" to the whole class of systems. Algebra (3.4) has an obvious Casimir function $\mathbf{S}^2 = \sum_{a=1}^3 S_j^a S_j^a$, which does not depend on the number of the site, $j = 1, \dots, N$ for the homogenous spin chain. In similar manner, one can define the classical analogs of the operators q_k and the hamiltonian $\mathcal{H}_N$.

The equations of motion of the classical spin magnet defined in this way look like

$$\frac{\partial z_n}{\partial t} = \{z_n, \mathcal{H}_N\} = \frac{\partial \mathcal{H}_N}{\partial p_n}\,, \qquad \frac{\partial p_n}{\partial t} = \{p_n, \mathcal{H}_N\} = -\frac{\partial \mathcal{H}_N}{\partial z_n}\,. \tag{3.5}$$

They possess the set of $N-1$ conserved charges q_n

$$\partial_t q_n = \{q_n, \mathcal{H}_N\} = 0\,. \tag{3.6}$$

with $n = 2, \dots, N$ and their solutions define the reggeon trajectories $z_k = z_k(t)$ subject to the periodicity condition $z_{k+N}(t) = z_k(t)$ and $p_{k+N}(t) = p_k(t)$.

To demonstrate a complete integrability of the evolution equations, one observes that the system (3.5) is equivalent to the matrix Lax pair relation

$$\partial_t L_k(x) = \{L_k(x), \mathcal{H}_N\} = A_{k+1}(x) L_k(x) - L_k(x) A_k(x)\,, \tag{3.7}$$

where $A_k(x)$ is a 2×2 matrix depending on the momenta and the coordinates of the particles. Let us introduce the Baker-Akhiezer function $\Psi_k(x;t)$ as a solution of the following system of matrix relations

$$L_k(x)\,\Psi_k(x,t) = \Psi_{k+1}(x,t)\,, \qquad \partial_t \Psi_k(x,t) = A_k(x)\,\Psi_k(x,t)\,. \tag{3.8}$$

The two-component Baker-Akhiezer function defined in this way depends on the site number k. The Lax operator $L_k(x)$ shifts it into the neighbouring site. One can introduce the monodromy operator producing the shift of Ψ_k along the whole chain as

$$\Psi_{k+N}(x;t) = T_N(x)\Psi_k(x;t)\,, \qquad T_N(x) \equiv L_N(x)\dots L_1(x)\,. \tag{3.9}$$

The periodic boundary conditions on the solutions to (3.5) can be easily formulated in terms of the Baker-Akhiezer function as

$$\Psi_{k+N}(x;t) = w\,\Psi_k(x;t)\,, \tag{3.10}$$

where w is the Bloch-Floquet factor. According to (3.9) and (3.10), w is the eigenvalue of the monodromy operator $T_N(x)$, and therefore it satisfies the characteristic equation

$$\det(T_N(x) - w\cdot\mathbf{1}) = 0 \tag{3.11}$$

From eqs. (3.7) and (3.9), the time dependence of the monodromy operator is given by $\partial_t T_N(x) = A_1(x)T_N(x) - T_N(x)A_1(x)$, and, as a consequence, its eigenvalues w are conserved quantities generating a complete set of integrals of motion. Evaluating the l.h.s. of (3.11), one obtains the spectral curve equation

$$w^2 - w\cdot \mathrm{Tr}\,T_N(x) + \det T(x) = 0\,. \tag{3.12}$$

Using the definitions (3.8) and (3.1) one gets

$$t_N(u) \equiv \mathrm{Tr}\,T_N(x) = 2x^N + q_2x^{N-2} + \cdots + q_{N-1}x + q_N \tag{3.13}$$

with q_k being the integrals of motion, eq. (3.2), and

$$\det T(x) = \prod_{k=1}^{N} \det L_k(x) = x^{2N}\,. \tag{3.14}$$

Introducing the complex function $y(x) = w - x^{2N}/w$ one obtains the equation of the spectral curve in the form

$$\Gamma_N:\ y^2 = t_N^2(x) - 4x^{2N} = \left(4x^N + q_2x^{N-2} + \cdots + q_N\right)\left(q_2x^{N-2} + \cdots + q_N\right). \tag{3.15}$$

For any complex x in general position the equation (3.12) has two solutions for w, or equivalently $y(x)$ in (3.15).

Under appropriate boundary conditions on $\Psi_k(x)$ these solutions define two branches of the Baker-Akhiezer function, $\Psi_k^{\pm}(x)$. Then, being a double-valued function on the complex x-plane, $\Psi_k(x)$ becomes a single-valued function on the Riemann surface corresponding to the complex curve Γ_N. This surface is constructed by gluing together two copies of the complex x-plane along the cuts $[e_2, e_3]$, ... , $[e_{2N-2}, e_1]$ running between the branching points e_j of the curve (3.15), defined as simple roots of the equation

$$t_N^2(e_j) = 4e_j^{2N}\,, \qquad j = 1, \ldots, 2N-2\,. \tag{3.16}$$

Here positions on the complex plane depend on the values of the integrals of motion q_2, q_3, ... , q_N. In general, the Riemann surface defined in this way has a genus $g = N-2$ which depends on the number of reggeons, N. For instance, it is a sphere at $N = 2$ and a torus at $N = 3$.

Applying the standard methods of the finite-gap theory, it becomes possible to construct the explicit expression for the Baker-Akhiezer function in terms of the theta-functions defined on the Riemann surface (3.15) and, finally, obtain the solutions to the classical equations of motion (3.5). The explicit expressions can be found in [12]. So far we restricted our consideration to the classical dynamics in the $z-$sector. Obviously, similar relations hold in the $\bar{z}-$sector. Since the integrals of motion, q_k and $\bar{q}_k$, and the hamiltonians in two sectors, H_N and $\bar{H}_N$, are conjugated to each other, the solutions of the classical equations of motion in two sectors can be obtained one from another.

Combining together the classical motion along the $z-$ and $\bar{z}-$directions, we find that the resulting expressions describe the finite-gap soliton wave propagating in the system of N particles located on the two-dimensional plane. In terms of the spectral curve (3.15), the same classical dynamics corresponds to the motion of particles on the Riemann surface (3.15). The motion on the Riemann surface is not confined to the kinematically allowed bands corresponding to the $\alpha-$cycles and the classical trajectories wrap along both the $\alpha-$ and $\beta-$cycles. This is one of the manifestation of the fact that we are dealing with two-dimensional system. As we see below, it will play an important rôle in our analysis.

3.2 Quantum case

Let us now consider the quantum evolution of the system with the hamiltonian defined in (2.15). As was explained in the section 2.1, the eigenstates of the noncompact $\mathrm{SL}(2,\mathbb{C})$ Heisenberg chain of length N and spin $(s = 0, \bar{s} = 1)$ play the role of the wave functions of the colourless compound states of N reggeized gluons. In addition, the ground state of the system controls the high-energy asymptotics of the scattering amplitudes, eq. (2.5).

Due to complete integrability of the system, the N-reggeon states have to diagonalize simultaneously the integrals of motion q_k and $\bar{q}_k$ with $k = 2, \ldots, N$ Their corresponding eigenvalues provide the total set of the quantum numbers of the N-reggeon states, $(q, \bar{q})$. In particular, the "lowest" integrals of motion, q_2 and $\bar{q}_2$, coincide with the quadratic Casimir operators for the total $\mathrm{SL}(2,\mathbb{C})$ spin of the system and their eigenvalues are related to the conformal $\mathrm{SL}(2,\mathbb{C})$ spin of the N-reggeon state as

$$q_2 = -h(h-1)\,, \qquad \bar{q}_2 = -\bar{h}(\bar{h}-1)\,, \qquad \bar{q}_2 = q_2^*\,, \tag{3.17}$$

where all possible values of h and $\bar{h} = 1 - h^*$ can be parameterized by integer n and real ν

$$h = \frac{1+n}{2} + i\nu\,, \qquad n = \mathbb{Z}\,, \quad \nu = \mathbb{R}\,. \tag{3.18}$$

As to remaining charges, $q_3, \ldots, q_N$ and $\bar{q}_k = q_k^*$, their possible values are also quantized. The explicit form of the corresponding quantization conditions is more complicated and it was obtained in [20].

According to (2.13), the integrals of motion are given by the $k-$order differential operators (with $k = 2, \ldots, N$) acting on the holomorphic and antiholomorphic reggeon coordinates. Therefore, their spectral problems leads to a complicated system of coupled differential equations on the wave functions of the N-reggeon states $\Psi_{q,\bar{q}}(\vec{z}_1, \ldots, \vec{z}_N)$. The solution to this system can be found by applying the method of separated variables developed by Sklyanin [28]. Namely, there exists the unitary transformation $U_{\vec{p},\vec{x}}(\vec{z})$

$$U_{\vec{p},\vec{x}_1,\ldots,\vec{x}_{N-1}}(\vec{z}_1,\ldots,\vec{z}_N) = \langle \vec{z}_1,\ldots,\vec{z}_N|\vec{p},\vec{x}_1,\ldots,\vec{x}_{N-1}\rangle\,, \tag{3.19}$$

that allows us to transform the wave function from the original $\vec{z}-$representation defined by the N coordinates of reggeons on the two-dimensional plane to the representation of the separated coordinates $\vec{x} = (\vec{x}_1,\ldots,\vec{x}_{N-1})$ and $\vec{p}$ being the total momentum of the system. The eigenstate of the hamiltonian in this representation is given by

$$\begin{aligned}\Phi_{\{q,\bar{q}\}}(\vec{x}_1,\ldots,\vec{x}_{N-1})\delta(\vec{p}-\vec{p}\,') &= \langle \Psi_{\vec{p},\{q,\bar{q}\}}|\vec{p}\,',\vec{x}_1,\ldots,\vec{x}_{N-1}\rangle \\ &= \int d^2z\, U_{\vec{p}\,',\vec{x}_1,\ldots,\vec{x}_{N-1}}(\vec{z})\,\Psi_{\vec{p},\{q,\bar{q}\}}(\vec{z})\,,\end{aligned} \tag{3.20}$$

where $d^2z = d^2z_1\cdots d^2z_N$. The remarkable feature of the unitary transformation $U_{\vec{p},\vec{x}}(\vec{z})$ is that it transforms the wave function $\Psi_{\vec{p},\{q,\bar{q}\}}(\vec{z})$ satisfying the original Schrödinger equation (2.6) into the wave function in the separated coordinates, $\Phi_{\{q,\bar{q}\}}(\vec{x})$, which is factorized into the product of $Q-$functions depending on a single separated coordinate $\vec{x}_k$ and satisfying the Baxter equation (see eqs. (3.23) and (3.24) below)

$$\Phi_{\{q,\bar{q}\}}(\vec{x}_1,\ldots,\vec{x}_{N-1}) = Q(x_1,\bar{x}_1)\ldots Q(x_{N-1},\bar{x}_{N-1})\,. \tag{3.21}$$

Here, the notation was introduced for the holomorphic and antiholomorphic components of the separated coordinates $\vec{x}_k = (x_k,\bar{x}_k)$. The explicit construction of the transformation to the separated coordinates, eq. (3.20), can be found in [33]. Requiring $U_{\vec{p}\,',\vec{x}_1,\ldots,\vec{x}_{N-1}}(\vec{z})$ to be a single-valued function on the $\vec{z}-$plane, one finds that the possible values of the separated coordinates $\vec{x}_k$ have the following form

$$x_k = \nu_k - \frac{in_k}{2}\,, \qquad \bar{x}_k = \nu_k + \frac{in_k}{2} \tag{3.22}$$

with ν_k real and n_k integer.

The functions $Q(x,\bar{x})$ entering (3.21) have to satisfy the functional Baxter equations in the x-sector

$$x^N Q(x+i,\bar{x}) + x^N Q(x-i,\bar{x}) = t_N(x)Q(x,\bar{x}) \tag{3.23}$$

and similar relation holds in the $\bar{x}-$sector

$$(\bar{x}+i)^N Q(x+i,\bar{x}) + (\bar{x}-i)^N Q(x-i,\bar{x}) = \bar{t}_N(\bar{x})Q(x,\bar{x})\,. \tag{3.24}$$

Here, $t_N(x)$ is a polynomial of degree N in x with the coefficients given by the eigenvalues of the integrals of motion, eq. (3.13). $\bar{t}_N(\bar{x})$ is given by similar expression with q_k replaced by $\bar{q}_k = q_k^*$. To make the correspondence with the classical mechanics, one introduces the shift operator on the space of functions $Q(x,\bar{x})$

$$\mathrm{e}^{\pm p}\,Q(x,\bar{x}) = Q(x\pm i,\bar{x})\,, \qquad \mathrm{e}^{\pm\bar{p}}\,Q(x,\bar{x}) = Q(x,\bar{x}\pm i)\,. \tag{3.25}$$

Then, the Baxter equations can be rewritten as

$$\left[x^N\left(\mathrm{e}^{p}+\mathrm{e}^{-p}\right) - t_N(x)\right]Q(x,\bar{x}) = \left[\bar{x}^N\left(\mathrm{e}^{\bar{p}}+\mathrm{e}^{-\bar{p}}\right) - \bar{t}_N(\bar{x})\right]\bar{x}^N Q(x,\bar{x}) = 0\,. \tag{3.26}$$

Based on the Baxter equations, it is now possible to establish the correspondence between the quantum and the classical systems. Let us forget for a moment that p and x are operators in (3.26) and treat them as $c-$numbers. If one identifies the Bloch-Floquet factor in (3.12) as $w(x) = x^N \exp(p)$, then it is easy to see that the l.h.s. of (3.26) coincides with the equation of spectral curve (3.12). Going back from the classical to quantum system, one concludes that the Baxter equation (3.26) arises in result of the "quantization" of the spectral curve (3.12) on the space of the function $Q(x, \bar{x})$. In other words, the constraint imposed by the spectral curve on the phase space of the classical system becomes an operator equation on the wave function of the same system after quantization. Thus the Riemann surface naturally enters into the quantum spectral problem. The important difference with respect to the classical case is that now the integrals of motion defining the moduli of the complex structure on (3.12) must be quantized.

The Baxter equations do not fix the function $Q(x, \bar{x})$ uniquely as their solutions are defined up to multiplication by an arbitrary periodic function $f(x+i, \bar{x}) = f(x, \bar{x}+i) = f(x, \bar{x})$. To avoid this ambiguity one has to impose additional conditions on the solutions to (3.23) and (3.24). As was demonstrated in [33, 20], these conditions follow from the explicit expression for the kernel of the unitary transformation (3.20) and they can be formulated as requirement for $Q(x, \bar{x})$ to have correct analytical properties and asymptotic behaviour at infinity. Namely, choosing $x = \lambda - in/2$ and $\bar{x} = \lambda + in/2$ in accordance with (3.22), and extending the values of λ from real axis to the complex plane, one requires that $Q_{q,\bar{q}}(x, \bar{x})$ should be a meromorphic function of λ for fixed integer n with an infinite set of poles of the order not higher than N situated at the points

$$\{x_m^{\pm} = \mp i m \,,\ \bar{x}_{\bar{m}}^{\pm} = \mp i(\bar{m} - 1)\}\,, \qquad m, \bar{m} = 1, 2, \ldots \tag{3.27}$$

with m and $\bar{m}$ being positive integer. In addition, the asymptotic behaviour of the solutions to the Baxter equation at infinity is given by

$$Q(\lambda - in/2, \lambda + in/2) \overset{\lambda\to\infty}{\sim} \mathrm{e}^{i\Theta}\, \lambda^{h+\bar{h}-N} + \mathrm{e}^{-i\Theta}\, \lambda^{1-h+1-\bar{h}-N} \tag{3.28}$$

with the phase $\Theta = \Theta_h(q, \bar{q})$ depending on the quantum numbers of the state and the total $\mathrm{SL}(2,\mathbb{C})$ spins h and $\bar{h} = 1 - h^*$ defined in (3.17). Up to an overall normalization, the above two conditions fix uniquely the solutions to the Baxter equation, and, in addition, they allow us to determine the spectrum of quantized integrals of motion q_k and $\bar{q}_k$.

Using the solution to the Baxter equations, we construct the wave function of the N-reggeon states in the separated coordinates (3.21). To find the corresponding value of the energy one has to apply the resulting expression for the wave function to the hamiltonian $\mathcal{H}_N$ transformed to the separated coordinates. This leads to the following remarkably simple expression for the energy of the N-reggeon state

$$\mathcal{H}_N = i \frac{d}{dx} \ln \left(x^{2N} \left[Q(x-i, x) \right]^* Q(x+i, x) \right) \Big|_{x=0} . \tag{3.29}$$

The relations (3.23), (3.24), (3.27), (3.28) and (3.29) provide the basis for calculating the spectrum of the N-reggeon states.

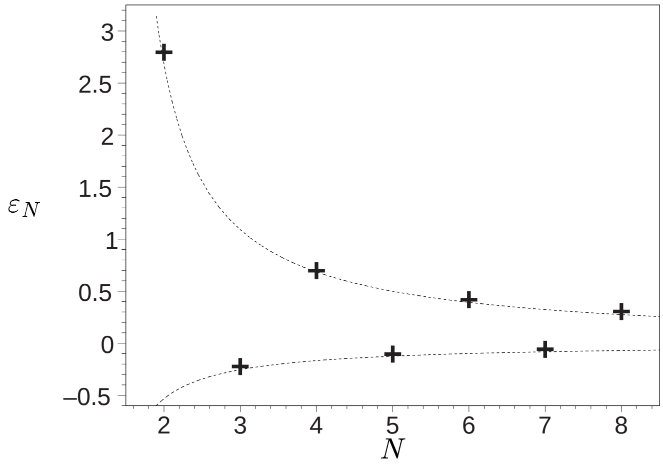

Figure 2: The dependence of the energy of the N-reggeon states, $E_N^{\max} = \alpha_s N_c \varepsilon_N / \pi$, on the number of particles N. The exact values of the energy are denoted by crosses. The upper and the lower dashed curves stand for the functions $1.8402/(N-1.3143)$ and $-2.0594/(N-1.0877)$, respectively.

The Baxter equations can be solved exactly at $N=2$ and their solution is expressed in terms of the ${}_3F_2-$hypergeometric series. Going though the calculation of the energy we arrive at

$$E_2(h) = \frac{\alpha_s N_c}{\pi}\epsilon_2 = -2\frac{\alpha_s N_c}{\pi}\,\mathrm{Re}\left[\psi\left(\frac{1+|n|}{2}+i\nu\right)-\psi(1)\right] . \tag{3.30}$$

This relation coincides with the well-known expression [36] for the energy of the $N=2$ reggeon compound state. The maximum value of the energy

$$E_2^{\max} = \frac{\alpha_s N_c}{\pi} 4\ln 2 \tag{3.31}$$

defines the intercept of the BFKL Pomeron. For $N=3$ the solution to the Schrödinger equation has been first found in [32] by direct diagonalization of the integral of motion q_3 in the original $\vec{z}-$coordinates. Recently, the same result was obtained within the Baxter equation approach in [34, 20]. The ground state of the $N=3$ system defines the Odderon state in QCD [31] and its energy is given by

$$E_3^{\max} = \frac{\alpha_s N_c}{\pi}\epsilon_3 = -\frac{\alpha_s N_c}{\pi}\cdot 0.247170\ldots \tag{3.32}$$

Contrary to (3.31), the energy of the $N=3$ state is negative. This implies that its contribution to the scattering amplitude (2.5) decreases at high-energy as $s\to\infty$, or $x\to 0$. Therefore, it becomes of most interest to find the solution of the Baxter equations, eqs. (3.23) and (3.24), for higher $N\geq 4$ reggeon states and calculate the spectrum of the energies ε_N governing the high-energy asymptotics in (2.5). This problem has been recently solved in [33, 20] and the results for ε_N are shown in the figure 2. The detailed description of the spectrum can be found in [20].

It was found in [20] that the N-reggeon states have different properties for even and odd N. For even N, the ground state of the system is *unique*, whereas for odd N it is *double degenerate*. In the latter case, the color-singlet compound states built odd number of reggeized gluons have the same energy but different quantum numbers with respect to the charge conjugation. For odd N the N-reggeon states with the charge parity $C = 1$ and $C = -1$ belong to the Pomeron and the Odderon sector, respectively. For even N, the reggeon states have the charge parity $C = 1$.

For *odd* N the ground state energy, ε_N, is *negative* and it increases with N approaching the value $\varepsilon_{2\infty+1} = 0$ from below. For *even* N, the ground state energy is *positive* and it decreases with N approaching the same value $\varepsilon_{2\infty} = 0$ from above. We recall that for $\varepsilon_N > 0$ ($\varepsilon_N < 0$) the contribution of the N-reggeon state to the cross-section (2.5) increases (decreases) at high energy s. Thus, in the Pomeron and the Odderon sectors, the contribution of the N-reggeon states to the cross-section ceases to depend on the energy s as $N \to \infty$. It is interesting to notice that this result has been anticipated a long time ago within the bootstrap approach [35].

As we have seen in the previous section, the classical analogs of the N-reggeon states are described by the finite-gap soliton waves propagating in the system of N particles on the plane. The solitons are uniquely specified by the Riemann surface (3.15) whose moduli are defined by the conserved charges q_k. The solution to the Schrödinger equation for the N-reggeon states leads to the quantization of these charges. This allows us to interpret the whole quantization procedure described in this section as quantization of the moduli space of the Riemann surface (3.15).

4. Universality class of the Regge limit

In the previous section the Riemann surfaces appeared within the context of high-energy QCD as auxiliary objects which were introduced to solve the equations of motion of the classical system of N reggeized gluons and the Baxter equations for the corresponding quantum system. In this section we shall argue that these surfaces have a definite physical meaning. Actually the situation is parallel to SUSY YM case where the auxiliary Riemann surface later on was identified as a part of the six-dimensional worldvolume of the M5-brane. The remaining four dimensions provide the worldvolume of the theory under consideration.

Here we shall develop a similar brane picture for the Regge limit of multi-colour QCD. To do this we explore the analogy with the $\mathcal{N} = 2$ SUSY YM where the stringy/brane picture naturally emerges from the hidden integrability governing the low-energy effective actions. We shall argue that in the Regge limit the Riemann surface which has been considered earlier plays a similar role providing the world-volume to the membrane describing the scattering process. In this way this Riemann surface fixes the universality class of the multi-colour QCD in the Regge limit.

4.1 Riemann surfaces and QCD versus SUSY YM

Let us recall the main features of the low-energy effective action in the $\mathcal{N} = 2$ SUSY YM theories relevant for our purposes [7, 37]. The key point is that the theory has the nontrivial vacuum manifold since the potential involves the term

$$V(\phi) = \mathrm{Tr}[\phi, \phi^{+}]^2 \,. \tag{4.1}$$

Here ϕ is the complex scalar field which generically develops the vacuum expectation value

$$\phi = \mathrm{diag}(\phi_1, \ldots, \phi_{N_c}) \,, \tag{4.2}$$

with $\mathrm{Tr}\,\phi = 0$. The gauge invariant order parameters $u_k = \langle 0 | \mathrm{Tr}\,\phi^k | 0 \rangle$ parameterize the Coulomb branch of the vacuum manifold. They define a scale in the theory with respect to which one can discuss the issue of a low energy effective action. This action takes into account one-loop perturbative correction as well as the whole instanton series and it is governed by the Riemann surface of the genus $N_c - 1$ for $\mathrm{SU}(N_c)$ gauge group. The same Riemann surface appears as the spectral curve of a classical integrable many-body system (see [38] for a review) which turns out to be the Calogero type systems or some spin chains. The period matrix $\tau_{ij}(u_k)$ of this Riemann surface yields the effective coupling constants of the gauge theory.

For example, in the SU(2) case one has

$$\tau_{\mathrm{eff}}(u_2) = i\frac{4\pi}{g^2(u_2)} + \frac{\theta(u_2)}{2\pi} \tag{4.3}$$

where $g^2(u_2)$ and $\theta(u_2)$ are the effective coupling and θ-term respectively. The spectrum of the BPS states in this theory is given by

$$M_{nm} = |na(u_2) + ma_D(u_2)| \,, \tag{4.4}$$

where a and a_D are the periods of some meromorphic differential λ_{SW} on the spectral curve Σ

$$a = \oint_A \lambda_{\mathrm{SW}} \,, \qquad a_D = \oint_B \lambda_{\mathrm{SW}} \,, \tag{4.5}$$

which in this case is a torus

$$\Sigma_{\mathrm{SU}(2)}: \qquad y^2 = (x^2 - \Lambda^4)(x - u_2) \,. \tag{4.6}$$

The prepotential $\mathcal{F}$ defining the low energy effective action is determined by the relations

$$\tau_{\mathrm{eff}}(u_2) = \frac{\partial a_D}{\partial a} = \frac{\partial^2 \mathcal{F}}{\partial^2 a} \,, \qquad a_D = \frac{\partial \mathcal{F}}{\partial a} \tag{4.7}$$

with $\tau_{\mathrm{eff}}(u_2)$ given by (4.3).

The connection with integrable system emerges when one studies the dependence of the prepotential on the fundamental scale Λ [39]

$$\frac{\partial \mathcal{F}(u_2)}{\partial \ln \Lambda} = \beta u_2 \equiv \beta H \,, \tag{4.8}$$

where β is the one-loop beta-function of the gauge theory. This equation has another interpretation as evolution equation of an integrable system where H coincides with the hamiltonian of this system and $\ln \Lambda$ is the evolution time variable. The integrable system provides the natural explanation for the appearance of the meromorphic differential λ_{SW}, which turns out to be the action differential in the separated variables $\lambda_{\mathrm{SW}} = p\,dx$. Let

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us emphasize that for SUSY YM case the *classical* integrable system is relevant and the meaning of the quantum system and the corresponding spectrum for SUSY YM remains an open question. It should involve the quantization of the vacuum moduli space since the hamiltonian in the dynamical system is nothing but $H = u_2 = \langle \mathrm{Tr}\, \Phi^2 \rangle$. Simultaneously, this parameter serves as the coordinate on the moduli space of the complex structures of the Riemann surfaces whiich means that the quantization of the integrable system is related with the quantization of the effective d=2 gravity.

The Riemann surface Σ degenerates at some points on the Coulumb branch on the moduli space. After the soft breaking of $\mathcal{N} = 2$ SUSY down to $\mathcal{N} = 1$ SUSY these points correspond to the vacuum states of the $\mathcal{N} = 1$ gauge theory. Some massless states condense at these points leading to the formation of a mass gap and to confinement. In the SU(2) case, eq. (4.6), the $\mathcal{N} = 1$ vacua correspond to the points $u_2 = \pm\Lambda^2$ at which the monopoles or dyons become massless and condense.

Let us now consider a particular theory, namely the superconformal $\mathcal{N} = 2$ SUSY YM with $N_f = 2N_c$ massless fundamental hypermultiplets [37, 40, 41]. The corresponding integrable system is described by the spectral curve Σ_{N_c}

$$y^2 = P^2_{N_c}(x) - 4x^{2N_c}(1 - \rho^2(\tau_{cl}))\,, \tag{4.9}$$

where $\rho^2(\tau_{cl})$ is some function of a coupling constant of the theory (see eq. (4.15) below) and the polynomial P_{N_c} depends on the coordinates on the moduli space $\vec{u} = (u_2, \dots, u_{N_c})$

$$P_{N_c}(x) = \sum_{k=0}^{N_c} q_k(\vec{u})\, x^{N_c-k} = 2x^{N_c} + q_2\, x^{N_c-2} + \cdots + q_{N_c}\,, \tag{4.10}$$

where $q_0 = 2$, $q_1 = 0$ and other q_k are some known functions of $\vec{u}$. Their explicit form is not important for our purposes. The Seiberg-Witten meromorphic differential on the curve (4.9) is given by

$$\lambda_{\rm SW} = p\, dx = \ln(\omega/x^{N_c})\, dx\,, \tag{4.11}$$

where $y = \omega - x^{2N_c}/\omega$.

From the point of view of integrable models the spectral curve (4.9) corresponds to a classical XXX Heisenberg spin chain of length N_c with the spin zero at all sites (due to $q_1 = 0$) and parameter ρ related to the external magnetic field, or equivalently, to the twisted boundary conditions [10].

Let us compare the spectral curve (4.9) for the superconformal $\mathcal{N} = 2$ SUSY YM with $N_f = 2N_c$ with the spectral curve (3.15) for N-reggeon compound states in multi-colour QCD. It is amusing to observe that these two curves coincide if we make the following identification

- The number of the reggeons $N = N_c$;
- The integrals of motion of multi-reggeon system are identified as the above mentioned functions $q_k(\vec{u})$ on the moduli space of the superconformal theory;
- The coupling constant of the gauge theory should be such that $\rho(\tau_{cl}) = 0$.

Under these three conditions the both theories fall into the same universality class.

Let us clarify the meaning of the last condition. Since the superconformal theory has a vanishing $\beta-$function one can assign a definite value to the bare coupling constant $\tau_{\rm cl}$. The function $\rho(\tau)$ entering (4.9) at $\tau=\tau_{\rm cl}$ is given by some combination of the modular forms [19, 18] which can be determined from the duality properties of the superconformal $\mathrm{SU}(N_c)$ gauge theory. It turns out that the duality groups are different for odd and even N_c. The group for even N_c is generated by

$$T: \quad \tau\to\tau+1, \qquad S: \quad \tau\to -1/\tau\,. \tag{4.12}$$

The group for odd N_c has the same T-transformation and

$$S: \quad \tau\to -\frac{1}{(2\cos(\pi/2N_c))^2\tau} \tag{4.13}$$

subject to the constaints

$$S^2=1 \qquad (ST^{-1})^{2N_c}=1 \tag{4.14}$$

The explicit formulae for $\rho(\tau)$ can be expressed in terms of weight four automorphic forms [19] for the duality group

$$\rho(\tau)=\frac{G_4(\tau)+H_4(\tau)}{G_4(\tau)-H_4(\tau)}\,, \tag{4.15}$$

where $G_4(\tau)$ is the Poincare series

$$G_4(\tau)=\sum_{c,d}(c\tau+d)^{-4}\,, \qquad H_4(\tau)=\frac{1}{\alpha^2\tau^4}G_4\left(-\frac{1}{\alpha\tau}\right), \tag{4.16}$$

with $\alpha=1$ for even N_c and $\alpha=4\cos^2(\pi/(2N_c))$ for odd N_c. Here the summation goes over all pairs of integers (c,d) which occur in modular transformations $\tau\to(a\tau+b)/(c\tau+d)$ generated by T and STS^{-1}. Finally, using (4.15) one finds that the condition $\rho(\tau_{cl})=0$ is satisfied for

$$\tau_{\rm cl}=\frac{1}{2}+\frac{i}{2}\tan\frac{\pi}{2N_c}\,. \tag{4.17}$$

The moduli of this superconformal theory are defined by the vector of the scalar condensates $\vec{u}$ and bare coupling constant $\tau_{\rm cl}$. The spectral curve (4.9) degenerates when $\rho\to\infty$ or when the discriminant of the curve is zero. Let us consider simplest nontrivial case $N_c=3$ which on the QCD side corresponds to the Odderon ($N=3$) state. In this case the discriminant is given by

$$(1-f)f^3\left[f-\left(1+\frac{2q_2^3}{27q_3^2}\right)^2\right]q_3^{10}=0 \tag{4.18}$$

where $f=1-\rho^2$. The curve degenerates at four values of ρ^2 among which three $\rho^2=\infty$, 0 and 1 are universal and the last one depends on the moduli q_2 and q_3 on the Coulomb branch. The first one is a strong coupling point corresponding to $\tau_{\rm cl}=0$. The second one is the so-called strong coupling orbifold point in which $\tau_{\rm cl}$ is given by (4.17). It is this point on the moduli space which corresponds to the Regge limit of multi-colour QCD. Finally, the third one is a weak coupling point $\tau_{\rm cl} \to \infty$. For $N_c = 3$ the explicit expression for the function $\rho(\tau)$, eq. (4.15), is given in terms of the Dedekind $\eta-$function by [19]

$$\rho(\tau) = \frac{f_+(\tau)}{f_-(\tau)}, \qquad f_\pm(\tau) = \left(\frac{\eta^3(\tau)}{\eta(3\tau)}\right)^3 \pm \left(3\frac{\eta^3(3\tau)}{\eta(\tau)}\right)^3 \tag{4.19}$$

The strong coupling orbifold point $\rho(\tau_{\rm cl}) = 0$ describing the Odderon state in QCD occurs at

$$\tau_{\rm cl} = \frac{1}{2} + \frac{i}{2\sqrt{3}}\,. \tag{4.20}$$

Let us note this point becomes the Argyres-Douglas point when the moduli dependent singularity of the spectral curve $\rho^2(\tau) = 1 - \left[1 + 2q_2^3/(27q_3^2)\right]^2$ collides with the strong coupling orbifold point (4.20). This happens when either $q_2/q_3 = 0$, or

$$q_3 = \pm\frac{(-q_2)^{3/2}}{\sqrt{27}}\,. \tag{4.21}$$

Going over to the general case of the $\mathrm{SU}(N_c)$ gauge theory one finds [19, 18] that independently on $\vec{u}$, the spectral curve (4.9) has three universal singularities at $\rho^2(\tau_{\rm cl}) = \infty$, 0 and 1 which have the same interpretation as in the case $N_c = 3$.

Finally we would like to note that there is the following intriguing fact. In the case of $N_c = 2$ which corresponds on the QCD side to the BFKL Pomeron state, the effective coupling constant is given in the weak coupling regime by the expression

$$\tau_{\rm eff} = \tau_{\rm cl} + i\frac{4\ln 2}{\pi} + \sum_k c_k \cdot \mathrm{e}^{2i\pi k\tau_{\rm cl}}\,, \tag{4.22}$$

where the second term is due to a finite one-loop correction [42] and the rest is the sum of instanton contributions. It is amusing that this one-loop correction to the coupling constant

$$\frac{1}{g^2_{\rm eff}} = \frac{1}{g^2_{\rm cl}}\left[1 + \frac{g^2_{\rm cl}}{4\pi^2}4\ln 2\right] + \cdots \tag{4.23}$$

coincides with the expression for the intercept of the BFKL Pomeron (3.31) after one identifies bare coupling constant in the superconformal theory with the t'Hooft coupling constant in QCD.

It would be very interesting to understand a proper QCD interpretation of the nonzero values of ρ in (4.9). From the point of view of spin chains it corresponds to the external magnetic field, or equivalently to the twisted boundary conditions [10]. The natural conjecture for $\rho \neq 0$ to occur is to consider the compound states in multi-colour QCD in which one of the reggeized gluons is replaced by the pair of reggeized quark and antiquark [43]. Similar picture takes place in the evolution equations for three-quark baryonic light-cone operators which will be discussed in section 6. In that case we shall deal with a real $\mathrm{SL}(2,\mathbb{R})$ spin chain whose spectrum contains special eigenstates (with $q = 0$ in eq. (6.20) below) for which a pair of quarks effectively behaves as one spin site [4]. For generic values of ρ we can have a genus one curve even for a two particle system and the whole power of the duality group can be used. For example, the behaviour of the system near the strong coupling orbifold point $\rho = 0$ can be related to the dual system near the weak coupling point $\rho = 1$. Certainly this possibility of having $\rho \neq 0$ in QCD deserves further investigation.

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4.2 Brane picture for the Regge limit

Let us turn now to a stringy/brane picture for the Regge limit in multi-colour QCD.

To warm up we would like to recall the brane description of the low-energy dynamics of the $\mathcal{N}=2$ SUSY YM. In the IIA framework the pure gauge theory is defined on the worldvolume of N_c D4-branes with the coordinates $(x_0, x_1, x_2, x_3, x_6)$ stretched between two NS5-branes with the coordinates $(x_0, x_1, x_2, x_3, x_4, x_5)$ and displaced along the coordinate x_6 by an amount inversely proportional to the coupling constant, $\delta x_6 = 1/g^2$ [23]. The coordinates at which the D4-branes intersect with the (x_4, x_5) complex plane define the vacuum expectation value of the scalar fields. This picture agrees perfectly with the RG behaviour of the coupling constant and yields the correct beta function in the gauge theory. The Riemann surface Σ discussed above describes the vacuum state of the theory and the spectrum of the stable BPS states. The lifting to the M theory picture leads to emergence of a single M5-brane with the worldvolume $R^4 \times \Sigma$.

In our case, we also need to incorporate into this picture branes corresponding to the fundamental matter with $N_f = 2N_c$. There are two ways to do this: either using semi-infinite D4-branes lifted into M5-brane in the M theory, or using D6-branes which induce the nontrivial KK monopole background for the M2-brane wrapped on the Riemann surface[23]. As was shown in [10] in the latter case the resulting brane picture remains consistent with the integrable spin chain dynamics and we shall stick to this case.

The explicit metric of the KK background in the M theory involving (x_4, x_5, x_6, x_{10}) coordinates has the multi-Taub-NUT form

$$ds^2 = \frac{V}{4} d\vec{r}^2 + \frac{V^{-1}}{4}(d\tau + \vec{A} d\vec{r})^2 \,, \tag{4.24}$$

where $\vec{r} = (x_4, x_5, x_6)$, $\tau = x_{10}$ and $\vec{A}$ is the Dirac monopole potential. The magnetic charge comes from the nontrivial twisting of S^1 bundle over R^3. The function V behaves as

$$V = 1 + \sum_{i=1}^{i=N_f} \frac{1}{|\vec{r} - \vec{r_i}|} \,, \tag{4.25}$$

where $\vec{r_i} = (x_4^i, x_5^i, x_6^i)$ are the positions of six-branes. For a superconformal case one must have $x_4^i = x_5^i = 0$ and positions of sixbranes in x_6 direction are irrelevant.

Finally let us remind the brane interpretation of the BPS spectrum in the $\mathcal{N}=2$ SUSY YM. One way to realize these states is to consider the self-dual noncritical strings (which come from M2-branes ending on the surface) wrapped around the Riemann surface [21]. Since one has to keep some amount of SUSY the wrapping of the string should be along geodesics in some metric. This is equivalent to looking for the geodesics on ω plane (in IIB approach), where ω is the same as in eq. (3.12) and is related to $y = \exp(-x_6 + ix_5)$ and $x = x_4 + ix_5$ as $y = \omega - x^{2N_c}/\omega$. The metric on ω plane is defined in terms of Seiberg-Witten differential

$$ds^2 = \lambda_{\rm SW} \bar{\lambda}_{\rm SW} \tag{4.26}$$

where $\bar{\lambda}_{\rm SW} = \bar{p}\, d\bar{x} = (\lambda_{\rm SW})^*$. Closed geodesics in the $\omega-$plane can be immediately lifted to closed curves on the Riemann surface. Hence the existence of the BPS state is the

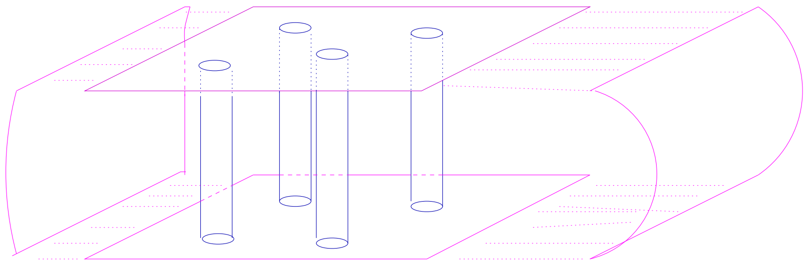

Figure 3: Spectral curve.

related to the existence of the closed geodesics in the $(n, m)-$cohomology class. Another important realization of BPS states arises from IIB/F theory picture. Namely the (n, m) BPS states come from the (n, m) strings which are stretched between the origin (which is generically split nonperturbatively into two strong coupling singularities) and the position of D3-brane we are living on [22]. Now the nontrivial metric on the Coulomb branch of the moduli space is involved. It has the form of the cosmic string metric with the conical singularity which can be brought into the flat form on the cover. In this picture $\mathrm{SL}(2, \mathbb{Z})$ invariance of the spectrum follows from the invariance of the IIB string theory.

Let us turn now to our proposal for the brane realization of the Regge limit. We shall explore the brane representations for the $N_f = 2N_c$ theory known in the IIA/M theory [23] and IIB/F theory [22]. However unlike the SUSY case where the spectral curve is embedded in the internal "momentum" space the spectral curve of the noncompact spin chain is placed in the phase space involving both impact parameter plane as well as momenta.

We shall first consider IIA/M type picture which is reminiscent to the realization of SYM theory via two NS5 and N_c D4-branes. We suggest that the coordinates involved in "IIA" picture are the transverse impact parameter coordinates x_1, x_2 and rapidity $\lambda = \ln k_+/k_-$. Transverse coordinates are analogue of (x_4, x_5) coordinates in SUSY case while rapidity substitutes the x_6 coordinate.

Now let us make the next step and suggest that similar to SUSY case the single brane is wrapped around the spectral curve of XXX magnet and two "hadronic planes" together with N "reggeonic strings" are just different projections of the single membrane with worldvolume $R \times \Sigma$. The coordinates involved into configurations are $x_1 + ix_2$ and $y = e^{-(\lambda + ix_{10})}$ where x_{10} is the "M-theory" coordinate. More precisely y can be identified with the Bloch-Floquet factor ω in the definition of the Baker-Akhiezer function (3.10). The correspondence between "IIA" and "M-theory" picture is shown in figure 3.

Let us emphasize once again that the brane configuration for Regge limit contrary to SYM case partially involves the coordinate space. More precisely the geometry of $N_f = 2N_c$ theory is determined by the following parameter [23]

$$\xi = -\frac{4\lambda_+\lambda_-}{(\lambda_+ - \lambda_-)^2}\,. \tag{4.27}$$

Here λ_+ and λ_- are asymptotic positions of five-branes defined by the large-x behaviour of the curve

$$\omega \propto \lambda_\pm x^{N_c}\,. \tag{4.28}$$

$\lambda_\pm$ can be found as roots of the equation

$$\lambda_\pm^2 + \lambda_\pm + \frac{1}{4}(1-\rho^2) = 0 \tag{4.29}$$

Since Regge limit corresponds to the strong coupling orbifold point, $\rho = 0$, the value of ξ is fixed as $\xi = \infty$. This corresponds to the coinciding branes at infinity.

Finally, the M theory brane picture for the Regge limit involves M5-brane corresponding to the vacuum state of the QCD. We can not say how it is placed precisely as the minimal surface in the internal seven dimensional space since the corresponding geometry is unknown yet. The new ingredient - membrane shares the time direction with the M5-brane and wrapping around the Riemann surface which is embedded into two-dimensional complex "phase space" with the multi-Taub-NUT metric determined by KK monopoles with the magnetic charge $2N$, which is a double number of reggeized gluons participating in the scattering process. The possible identification of the membrane above with the M2-brane deserves further investigation.

5. Quantum spectrum and S-duality

The S-duality is a powerful symmetry in the SUSY YM theory which allows us to connect the weak and strong coupling regimes. The effective coupling in this theory coincides with the modular parameter of the spectral curve of the underlying classical integrable model. As a consequence, the S-duality transformations in the gauge theory are translated into the modular transformations of the spectral curve describing complexified integrable system. A formulation of the S-duality in the latter system naturally leads to an introduction of the notion of the dual action [44].

So far the S-duality was well understood only for classical integrable models. In the case of multi-colour QCD in the Regge limit the situation is more complicated since the duality has to be formulated for a quantum integrable model. The integrals of motion take quantized set of values and the coordinates on the moduli space are not continuous anymore. Therefore the question to be answered is whether it is possible to formulate some duality transformations at the quantum level.

To study this question let us propose the WKB quantization conditions which are consistent with the duality properties of the complexified dynamical system whose solution to the classical equations of motion are described by the Riemann surface. We recall that the standard WKB quantization conditions involve the real slices of the spectral curve

$$\oint_{A_i} p\, dx = 2\pi\hbar\Big(n_i + \frac{1}{2}\Big)\,, \tag{5.1}$$

where n_i are integers and the cycles A_i correspond to classically allowed trajectories on the phase space of the system. In our case the coordinate x is complex and arbitary point on the Riemann surface is classically allowed. As a result the general classical motion involves both $A-$ and $B-$cycles on the Riemann surface. This leads to the following generalized WKB quantization conditions (see also [20])

$$\mathrm{Re}\oint_{A_i} p\, dx = \pi\hbar n_i\,, \qquad \mathrm{Re}\oint_{B_i} p\, dx = \pi\hbar m_i\,, \tag{5.2}$$

where the “action” differential was defined in (4.11). Note that in the context of the SUSY YM this condition would correspond to the nontrivial constraints on the periods (4.5) and on the mass spectrum of the BPS particles (4.4). It is clear that the WKB conditions (5.2) imply the duality $A_i \leftrightarrow B_i$ and $n_i \leftrightarrow m_i$.

The conditions (5.2) are analogous to the WKB quantization for a particle in a multi-dimensional case. We have exactly this situation since we are dealing with complexified phase space. For a particle in higher dimensions one considers the representation of the wave function in terms of the multivalued functions A_k and S_k on the $\vec{x}$-space

$$\Psi(\vec{x}) = \sum_k A_k(\vec{x}) \exp\left(\frac{i}{\hbar} S_k(\vec{x})\right) . \tag{5.3}$$

Here $A_k(\vec{x})$ and $S_k(\vec{x})$ are different branches of the multivalued functions $A(\vec{x})$ and $S(\vec{x})$. To obtain the WKB quantization conditions one considers the covering space on which $A(\vec{x})$ and $S(\vec{x})$ are single valued and requires that the wave function $\Psi(\vec{x})$ should resume its original value after encircling along all noncontructable cycles C_i on the covering manifold. This leads to

$$\oint_{C_i} \vec{p}\cdot d\vec{x} + \oint_{C_i} d\ln A(\vec{x}) = 2\pi\hbar n_i\,. \tag{5.4}$$

The second integral is equal to the sum of two terms related to the topological invariants of the classical trajectory: the number of intersections of C_i with hypersurface of the caustics and the number of intersections of C_i with the hypersurface of the turning points. In our case $\oint_{C_i} \vec{p}\cdot d\vec{x} = 2\,\mathrm{Re}\oint_{C_i} p\,dx$ and $\oint_{C_i} d\ln A(\vec{x})$ is an even integer.

Let us consider the quantization conditions (5.2) in the simple case of the Odderon $N = 3$ system. The spectral curve (3.15) is a torus

$$y^2 = (2x^3 + q_2 x + q_3)^2 - 4x^6\,, \tag{5.5}$$

where q_2 is given by (3.17) and (3.18) while q_3 is the complex integral of motion to be quantized. The quantization conditions (5.2) read (for $\hbar = 1$)

$$\mathrm{Re}\,a(q_3) = \pi n, \qquad \mathrm{Re}\,a_D(q_3) = \pi m\,, \tag{5.6}$$

where n and m are integer. These equations can be solved for large values of $q_3^2/q_2^3 \gg 1$, for which the expressions for the periods $a(q_3)$ and $a_D(q_3)$ are simplified considerably. The explicit evaluation of the integrals (4.5) in this limit yields

$$a(q_3) = \frac{(2\pi)^2 q_3^{1/3}}{\Gamma^3(2/3)}\,, \qquad a_D(q_3) = a(q_3)\left(\frac{1}{2} + \frac{i}{2\sqrt{3}}\right) . \tag{5.7}$$

Substituting these expressions into (5.6) one finds [20]

$$q_3^{1/3} = \frac{\Gamma^3(2/3)}{2\pi}\left(\frac{\ell_1}{2} + i\frac{\sqrt{3}}{2}\ell_2\right) . \tag{5.8}$$

where $\ell_1 = n$ and $\ell_2 = n - 2m$. The same quantization conditions can be also written as

$$(\ell_1 + 3\ell_2) \cdot a(q_3) - 6\ell_2 \cdot a_D(q_3) = \pi(\ell_1^2 + 3\ell_2^2)\,. \tag{5.9}$$

The WKB expressions (5.8) are in a good agreement with the exact expressions for quantized q_3 obtained from the solutions of the Baxter equations in [20].

There is a simple relation between the modular parameter of the curve with the periods a and a_D

$$a_D = \tau_{\text{eff}}\, a\,. \tag{5.10}$$

It follows from the fact that a and a_D are the solutions to the Picard-Fuchs equations which in the Odderon case is the second order differential equation. One can show that the Wronsian for this equation $aa'_D - a_D a'$ vanishes and using $\tau_{\text{eff}} = a'_D/a'$ one gets (5.10). Due to this fact half of quantization conditions can be formulated in terms τ_{eff} instead of a_D.

Let us emphasize that the point at the moduli space corresponding to the degeneration of the torus for the Odderon case does not appear in the quantum spectrum. From the point of view of the gauge theory this means that the appearance of the massless states is forbidden.

In the general multi-reggeon case we have to consider the quantization conditions (5.2) on the Riemann surface (3.15) of the genus $(N-2)$ which has the same number of the $A-$ and $B-$cycles. In result the spectrum of the integrals of motion $q_3, \ldots, q_N$ is parametrized by two $(N-2)-$component vectors $\vec{n}$ and $\vec{m}$. In the SUSY YM case these vectors define the electric and magnetic charges of the BPS states. In the Regge case the physical interpretation of $\vec{n}$ and $\vec{m}$ is much less evident. Let us first compare the electric quantum numbers in the two cases. In Regge case it corresponds to rotation in the coordinate space around the ends of the reggeons. This picture fits perfectly with the interpretation of the electric charge in SUSY YM case. Indeed VEVs of the complex scalar take values on the complex plane which is the counterpart of the impact parameter plane and the rotation of the phase of the complex scalar is indeed the "electric rotation".

To get some guess concerning the "magnetic" quantum numbers it is instructive to check the geometrical picture behind them in the simplified "IIA" picture. All states corresponding to the "electric" degrees of freedom are related to fundamental strings encircling "reggeoinic" tubes (see figure 3) and don't feel the hadronic planes. However the "magnetic" states as is well known from SUSY YM case are represented by the membrane stretched between two strings and two hadronic planes. Therefore these states are sensitive to hadronic quantum numbers. More detailed interpretation of "magnetic" degrees of freedom in the Regge regime has to be recovered.

Let us emphasize that due to the stringy realization of the abovementioned BPS spectrum we obtain rather transparent picture of the WKB quantization in the stringy terms. Remind that, in SUSY YM the mass of BPS state $M_{n,m}$ is given by the energy of the dyonic (n, m) string stretched between the value of u_2 corresponding to a given VEV of a scalar field and the point $u_2 = 0$ where orientifold and all D7-branes are placed. Hence,

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the WKB quantization at least in the Odderon case can be formulated as the quantization of the energy of the dyonic strings at the complex q_3 plane and the spectrum inherited the duality from S-duality of IIB string.

6. Stringy/brane picture and the calculation of the anomalous dimensions

In the previous sections we have demonstrated that integrability properties of the Schrödinger equation for the compound state of Reggeized gluons give rise to the stringy/brane picture for the Regge limit in multi-colour QCD. There is another limit in which QCD exhibit remarkable properties of integrability. It has to do with the scaling dependence of the structure functions of deep inelastic scattering and hadronic light-cone wave functions in QCD. In the both cases, the problem can be studied using the Operator Product Expansion and it can be reformulated as a problem of calculating the anomalous dimensions of the composite operators of a definite twist. The operators of the lowest twist have the following general form

$$\begin{aligned} \mathcal{O}^{(2)}_{N,k}(0) &= (yD)^k\Phi_1(0)(yD)^{N-k}\Phi_2(0)\,, \\ \mathcal{O}^{(3)}_{N,\boldsymbol{k}}(0) &= (yD)^{k_1}\Phi_1(0)(yD)^{k_2}\Phi_2(0)(yD)^{N-k_1-k_2}\Phi_3(0)\,, \end{aligned} \tag{6.1}$$

where $\boldsymbol{k} \equiv (k_1, k_2)$ denotes the set of integer indices k_i, y_μ is a light-cone vector such that $y_\mu^2 = 0$. Φ_k denotes elementary fields in the underlying gauge theory and $D_\mu = \partial_\mu - iA_\mu$ is a covariant derivative. The operators of a definite twist mix under renormalization with each other. In order to find their scaling dependence one has to diagonalize the corresponding matrix of the anomalous dimension and construct linear combination of such operators, the so-called conformal operators [45, 46]

$$\mathcal{O}^{\rm conf}_{N,q}(0) = \sum_{\boldsymbol{k}} C_{\boldsymbol{k},\boldsymbol{q}}\cdot\mathcal{O}_{N,\boldsymbol{k}}(0)\,. \tag{6.2}$$

A unique feature of these operators is that they have an autonomous RG evolution

$$\Lambda^2\frac{d}{d\Lambda^2}\,\mathcal{O}^{\rm conf}_{N,q}(0) = -\gamma_{N,q}\cdot\mathcal{O}^{\rm conf}_{N,q}(0)\,. \tag{6.3}$$

Here Λ^2 is a UV cut-off and $\gamma_{N,q}$ is the corresponding anomalous dimension depending on some set of quantum numbers $\boldsymbol{q}$ to be specified below. It turns out that the problem of calculating the spectrum of the anomalous dimensions $\gamma_{N,q}$ to one-loop accuracy becomes equivalent to solving the Schrödinger equation for the $\mathrm{SL}(2,\mathbb{R})$ Heisenberg spin magnet. The number of sites in the magnet is equal to the number of fields entering into the operators under consideration.

To explain this correspondence it becomes convenient to introduce nonlocal light-cone operators

$$F(z_1,z_2) = \Phi_1(z_1y)\Phi_2(z_2y)\,, \qquad F(z_1,z_2,z_3) = \Phi_1(z_1y)\Phi_2(z_2y)\Phi_3(z_3y)\,. \tag{6.4}$$

Here y_μ is a light-like vector ($y_\mu^2 = 0$) defining certain direction on the light-cone and the scalar variables z_i serve as a coordinates of the fields along this direction. The fields

$\Phi_i(z_i y)$ are transformed under the gauge transformations and it is tacitly assumed that the gauge invariance of the nonlocal operators $F(z_i)$ is restored by including the Wilson lines between the fields in the appropriate (fundamental or adjoint) representation. The conformal operators appear in the OPE expansion of the nonlocal operators (6.4) for small $z_1 - z_3$ and $z_2 - z_3$.

The field operators entering the definition of $F(z_i)$ are located on the light-cone. This leads to the appearance of the additional light-cone singularities. They modify the renormalization properties of the nonlocal light-cone operators (6.4) and lead to nontrivial evolution equations which as we will show below become related to integrable chain models. We notice that there exists the following relation between the conformal three-particle operators (6.2) and the nonlocal operators (6.4)

$$\mathcal{O}^{\rm conf}_{N,q}(0) = \Psi_{N,q}(\partial_{z_1}, \partial_{z_2}, \partial_{z_3}) F(z_1, z_2, z_3)\Big|_{z_i=0}\,, \tag{6.5}$$

where $\Psi_{N,q}(x_1, x_2, x_3)$ is a homogenous polynomial in x_i of degree N

$$\Psi_{N,q}(x_1, x_2, x_3) = \sum_{\boldsymbol{k}} C_{\boldsymbol{k},\boldsymbol{q}} \cdot x_1^{k_1} x_2^{k_2} x_3^{N-k_1-k_2} \tag{6.6}$$

with the expansion coefficients $C_{\boldsymbol{k},\boldsymbol{q}}$ defined in (6.2). Similar relations hold for the twist-2 operators. The problem of defining the conformal operators is reduced to finding the polynomial coefficient functions $\Psi_{N,q}(x_i)$ and the corresponding anomalous dimensions $\gamma_{N,q}$. Using the renormalization properties of the nonlocal light-cone operators (6.4) one can show [4], that to the one-loop accuracy the QCD evolution equation for the conformal operators (6.4) can be rewritten in the form of a Schrödinger equation

$$\mathcal{H} \cdot \Psi_{N,q}(x_i) = \gamma_{N,q} \Psi_{N,q}(x_i)\,, \tag{6.7}$$

where the hamiltonian $\mathcal{H}$ acts on the x_i-variables which are conjugated to the derivatives ∂_{z_i} and, therefore, have the meaning of light-cone projection $(y \cdot p_i)$ of the momenta p_i carried by particles described by fields $\Phi(z_i y)$.

The hamiltonian $\mathcal{H}$ has the following remarkable properties.

(i) This hamiltonian is a sum of two-particle hamiltonians

$$\mathcal{H}^{(2)} = H_{12}\,, \qquad \mathcal{H}^{(3)} = H_{12} + H_{23} + H_{13}\,, \tag{6.8}$$

where H_{ij} acts on the coordinates of particles i and j.

(ii) All two-particle hamiltonians H_{ij} are invariant under $\mathrm{SL}(2,R)$ transformations

$$[H_{ij}, \vec{S}_i + \vec{S}_j] = 0\,, \tag{6.9}$$

where one-particle $\mathrm{SL}(2,R)$ generators $\vec{S}_i$ act on the space of functions $\Psi(x_i)$ as

$$\begin{aligned} S_{k,0}\Psi(x_k) &= (x_k\partial_k + j_k)\,\Psi(x_k)\,, \\ S_{k,+}\Psi(x_k) &= -x_k\Psi(x_k)\,, \\ S_{k,-}\Psi(x_k) &= \left(x_k\partial_k^2 + 2j_k\partial_k\right)\Psi(x_k)\,. \end{aligned} \tag{6.10}$$

Let us note that these generators where x_k play the role of momenta variables are dual to generators (2.14) where z_k were coordinates. This symmetry of one-loop evolution kernels H_{ij} follows from the invariance of classical QCD lagrangian under the group of conformal transformations [45] which is reduced to its $\mathrm{SL}(2,R)$ subgroup on the light-cone

$$z \to z' = \frac{az+b}{cz+d}\,, \qquad \Phi_k(zy) \to \Phi_k'(z'y) = (cz+d)^{-2j_k}\Phi_k\left(\frac{az+b}{cz+d}y\right) \tag{6.11}$$

with $ad-bc=1$ and a,b,c,d real. Here

$$j_k = \frac{1}{2}(d_k+s_k)\,, \tag{6.12}$$

where d_k is a canonical dimension of a field $\Phi_k(zy)$ and s_k is a light-cone component of it spin. For example, $d_q=3/2$ and $s_q=1/2$ and $j_q=1$ for quarks. Due to (6.9) hamiltonians H_{ij} are functions of the two-particle Casimir operators

$$(\vec{S}_1+\vec{S}_2)^2 = J_{12}(J_{12}-1)\,. \tag{6.13}$$

For example when Φ_1 and Φ_2 are quark fields of the same chirality

$$F_{\alpha\beta}(z_1,z_2) = \sum_{i=1}^{N_c}(\bar{q}_i^{\uparrow}\not{y})_\alpha(z_1y)(\not{y}q_i^{\uparrow})_\beta(z_2y) \tag{6.14}$$

with $q_i^{\uparrow}=(1+\gamma_5)q_i/2$, the two-particle hamiltonian is given by [46]

$$H_{12} = \frac{\alpha_{\mathrm{s}}}{\pi}C_F\left[H_{qq}(J_{12})+\frac{1}{4}\right], \qquad H_{qq}(J_{12}) = \psi(J_{12})-\psi(2)\,. \tag{6.15}$$

where $C_F = (N_c^2-1)/(2N_c)$. The eigenfunctions for this hamiltonian are the highest weights of the discrete series representation of the $\mathrm{SL}(2,R)$ group

$$\Psi_N^{(2)}(x_1,x_2) = (x_1+x_2)^N C_N^{3/2}\left(\frac{x_1-x_2}{x_1+x_2}\right) \tag{6.16}$$

where $C_N^{3/2}$ are Gegenbauer polynomials. The corresponding eigenvalues define the anomalous dimensions of the twist-2 mesonic operators built from two quarks with the same helicity

$$\gamma_N^{(2)} = \frac{\alpha_{\mathrm{s}}}{\pi}C_F[\psi(N+2)-\psi(2)+1/4] = \frac{\alpha_{\mathrm{s}}}{\pi}C_F\left[\sum_{k=1}^{N}\frac{1}{k+1}+\frac{1}{4}\right]. \tag{6.17}$$

At large N this expression has well-known asymptotic behaviour $\gamma_N^{(2)} \sim \alpha_{\mathrm{s}}C_F/\pi\ln N$.

It is conformal symmetry which dictates that the two-particle hamiltonian is a function of the Casimir operator of the $\mathrm{SL}(2,\mathbb{R})$ group, but it does not fix this function. The fact that this function turns out to be the Euler ψ-function leads to a hidden integrability of the evolution equations for anomalous dimensions of baryonic operators in very much the same way as emergence of the Euler ψ-functions in the BFKL kernel (2.11) leads to the integrability of multi-colour QCD in the Regge limit. Namely, for baryonic operator built from three quark fields of the same chirality

$$F_{\alpha\beta\gamma}(z_1,z_2,z_3)=\sum_{i,j,k=1}^{N_c}\epsilon_{ijk}(\not{y}q_i^{\uparrow})_\alpha(z_1y)(\not{y}q_j^{\uparrow})_\beta(z_2y)(\not{y}q_k^{\uparrow})_\gamma(z_3y) \tag{6.18}$$

the evolution kernel is given by [3, 4]

$$\mathcal{H}^{(3)}=\frac{\alpha_s}{2\pi}\left\{(1+1/N_c)\left[H_{qq}(J_{12})+H_{qq}(J_{23})+H_{qq}(J_{31})\right]+3C_F/2\right\} \tag{6.19}$$

with H_{qq} given by (6.15). The Schrödinger equation (6.7) with the hamiltonian defined in this way has a hidden integral of motion

$$q=i\left(\partial_{x_1}-\partial_{x_2}\right)\left(\partial_{x_2}-\partial_{x_3}\right)\left(\partial_{x_3}-\partial_{x_1}\right)x_1x_2x_3 \tag{6.20}$$

and, therefore, it is completely integrable. This operator is dual to the operator q_3 for the $N=3-$reggeon states, eq. (2.13), in the same fashion as the SL(2) generators were related in the two cases. Similar to the Regge case, one can identify (6.19) as the hamiltonian of a quantum XXX Heisenberg magnet of $\mathrm{SL}(2,\mathbb{R})$ spin $j_q=1$. The number of sites is equal to the number of quarks.

Based on this identification we shall argue now that the calculation of the anomalous dimensions can be formulated entirely in terms of Riemann surfaces which in turn leads to a stringy/brane picture. It is important to stress here the key difference between Regge and light-cone limits of QCD. In the first case the impact parameter space provides the complex plane for the reggeon coordinates and we are dealing with a $(2+1)-$dimensional dynamical system. In the second case the QCD evolution occurs along the light-cone direction and is described by a $(1+1)-$dimensional dynamical system. As a consequence, in these two cases we have two different integrable magnets: the $\mathrm{SL}(2,\mathbb{C})$ magnet for the Regge limit and the $\mathrm{SL}(2,\mathbb{R})$ magnet for the light-cone limit. The evolution parameters ("time" in the dynamical models) are also different: the rapidity $\ln s$ for the Regge case and the RG scale $\ln\mu$ for the anomalous dimensions of the conformal operators.

Our approach to calculation of the anomalous dimensions via Riemann surfaces looks as follows. For concreteness, we shall concentrate on the evolution kernel (6.19). Similarly to the Regge case, one starts with the finite-gap solution to the classical equation of motion of the underlying $\mathrm{SL}(2,\mathbb{R})$ spin chain and identifies the corresponding Riemann surface

$$\omega-\frac{x^6}{\omega}=2x^3-(N+2)(N+3)x+q\,,\qquad \omega=x^3\,\mathrm{e}^p \tag{6.21}$$

where q is the above mentioned integral of motion (6.20) and N is the total $\mathrm{SL}(2,\mathbb{R})$ spin of the magnet, or equivalently the number of derivatives entering the definition of the conformal operator (6.2). In distinction with the $\mathrm{SL}(2,\mathbb{C})$ case, eq. (3.18), the total spin N takes nonnegative integer values. Note that the Riemann surface corresponding to the three-quark operator has genus $g=1$, while $g=0$ for the twist 2 operators.

At the next step we quantize the Riemann surface as follows. We replace $p = i\partial/\partial x$ and impose the equation of the complex curve as the operator annihilating the Baxter function

$$\left(\mathrm{e}^{i\partial/\partial x} + \mathrm{e}^{-i\partial/\partial x}\right) x^3 Q(x) = \left[2x^3 - (N+2)(N+3)x + q\right] Q(x)\,. \tag{6.22}$$

This leads to the Baxter equation for the Heisenberg $\mathrm{SL}(2,\mathbb{R})$ magnet of spin $j = 1$

$$(x+i)^3 Q(x+i) + (x-i)^3 Q(x-i) = \left[2x^3 - (N+2)(N+3)x + q\right] Q(x)\,. \tag{6.23}$$

Similar to the Baxter equation in the Regge case, this equation does not have a unique solution. To avoid this ambiguity one has to impose the additional conditions that $Q(x)$ should be polynomial in x. This requirement leads to the quantization of the integral of motion. The resulting polynomial solution $Q = Q_q(x)$ has the meaning of the one-particle wave function in the separated variables x which in the case of the $\mathrm{SL}(2,\mathbb{R})$ magnet take arbitrary real values.

Given the polynomial solution to the Baxter equation (6.23), one can determine the eigenspectrum of the hamiltonian (6.19) and in result the anomalous dimensions of the corresponding baryon operators

$$\gamma^{(3)}_{N,q} = \frac{\alpha_{\mathrm{s}}}{2\pi}\left[\left(1 + \frac{1}{N_c}\right)\mathcal{E}_{N,q} + \frac{3C_F}{2}\right], \qquad \mathcal{E}_{N,q} = i\frac{Q_q'(i)}{Q_q(i)} - i\frac{Q_q'(-i)}{Q_q(-i)} \tag{6.24}$$

parameterized by the eigenvalues of the integral of motion (6.20) given by

$$q = -i\frac{Q_q(i) - Q_q(-i)}{Q_q(0)}\,. \tag{6.25}$$

The solution of the Baxter equation simplifies greatly in the quasiclassical approximation which is controlled by the total $\mathrm{SL}(2,\mathbb{R})$ spin of the system N. For $N \gg 1$ the spectrum of the integral of motion q is determined by the WKB quantization condition [11, 13]

$$\oint_A p\, dx = 2\pi\left(n + \frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{N}\right), \tag{6.26}$$

where p was introduced in (6.22). Here integration goes over the A-cycle on the Riemann surface defined by the spectral curve (6.22). This cycle encircles the interval on the real x-axis on which $|\,\mathrm{e}^p\,| > 1$. Solving (6.26) one gets

$$q = \pm\frac{N^3}{\sqrt{27}}\left[1 - 3\left(n + \frac{1}{2}\right)N^{-1} + \mathcal{O}(N^{-2})\right]. \tag{6.27}$$

Comparing this expression with (4.21) and taking into account that in this case $q_2 = -(N+3)(N+2)$ and $q_3 = q$ we conclude that for $N \to \infty$ the system is approaching the Argyres-Douglas point. Note also that the WKB quantization conditions (6.26) involve only the $A-$cycle on the Riemann surface and unlike the Regge case there is no S-duality in the quantum spectrum.

Finally, the spectrum of the anomalous dimensions in the WKB approximation is given by [11, 13, 4]

$$\mathcal{E}_{N,q} = 2\ln 2 - 6 + 6\gamma_E + 2\mathrm{Re}\sum_{k=1}^{3}\psi(1+i\delta_k) + \mathcal{O}(N^{-6}), \tag{6.28}$$

where δ_k are defined as roots of the following cubic equation:

$$2\delta_k^3 - (N+2)(N+3)\delta_k + q = 0 \tag{6.29}$$

and q satisfies (6.27).

What can we learn about stringy picture from this information about anomalous dimensions? Let us remind that in the spirit of string/gauge fields correspondence [47] the anomalous dimensions of gauge field theory operators coincide with excitation energies of a string in some particular background. An important lesson that we have learned from the analysis of the the two- and three-quark operators is that in the first case the anomalous dimensions are uniquely specified by a *single* parameter N which define the total $\mathrm{SL}(2,\mathbb{R})$ spin. In the second case, a new quantum number emerges due to the fact that the corresponding dynamical system is completely integrable. The additional symmetry can be attributed to the operator q defined in (6.20). From the point of view of a classical dynamics this operator generates the winding of a particle around the $A-$cycles on the spectral curve. Within the string/gauge fields correspondence one expects to reproduce these properties using a description in terms of the same string propagating in different backgrounds. One is tempting to suggest that different properties of the anomalous dimensions of the two- and three-particle opereators should be attributed to different properties of the background. In the case of the twist-2 the anomalous dimensions depend on integer N which in the classical system has an interpretation of the total $\mathrm{SL}(2,\mathbb{R})$ angular momentum of the system. On the stringy side the same parameter has a natural interpretation as a string angular momentum.

When this paper was in a process of preparation the interesting paper concerning the stringy derivation of the anomalous dimensions of the twist-2 operators [48] appeared.[1] It was shown that the classical solution to the equation of motion in the σ-model on $AdS_5\times S^5$ provides the anomalous dimension of the twist-2 operators with the large spin N. Since our derivation of the same anomalous dimension is seemingly different it is very instructive to compare two approaches. The logic behind derivation in [48] implies that one considers the stringy σ model perturbed by some vertex operator representing the operator under consideration on the gauge theory side. Then the energy on the solution to the classical equations of motion in $\sigma-$model involving the radial coordinate in the bulk provides the anomalous dimension of the operator.

In our approach we have the following correspondence

$$\begin{aligned}
\text{operator} &\Longleftrightarrow \text{Riemann surface}\\
\text{twist of the operator} &\Longleftrightarrow \text{genus of the Riemann surface}\\
\text{calculation of the anomalous dimension} &\Longleftrightarrow \text{quantization of the Riemann surface}
\end{aligned}$$

[1]Note that some attempts to develop the σ-model representation for high energy QCD were undertaken a long time ago [49].

It seems that the Riemann surfaces whose moduli (the integrals of motion of the spin chain) define the anomalous dimensions of the corresponding operators describe the $\sigma-$model solutions found in [48]. The precise relation between two approaches need to be clarified further and will be discussed elsewhere.

As we have seen the quantization of the Riemann surface can be performed most effectively in terms of the Baxter equation. It is worth noting that the solution to the Baxter equation can be identified as a wave function of D0-brane [50]. Quantization conditions arise from the requirement for the wave function the D0-brane probe in the background of the Riemann surface to be well defined.

In the case of the light-cone composite operators we have to incorporate into the stringy picture a new quantum number which is parameterized by an integer n, eq. (6.27), i.e. string excitation spectrum has now different sectors parameterized by this integer. The natural way to interpret these sectors is to identify n with the winding number of a closed string. The corresponding background for such scenario is offered by the Riemann surface itself with the string wrapped around the $A-$cycle. It is an interesting open question if one can interpret a momentum in WKB quantization condition (6.26) as a momentum of a string T-dual to the string with the winding number n.

Since the spectrum of the anomalous dimensions in QCD coincides with the spectrum of the $\mathrm{SL}(2,\mathbb{R})$ spin chain hamiltonian it would be interesting to explore further the symbolic relation

$$H_{\text{string}} \propto H_{\text{spin}} \,, \tag{6.30}$$

where string propagates in the background determined by the Riemann surface of the spin chain. It is known that hamiltonian formulation of the spin chains is closely related to the Chern-Simons (CS) theory with the inserted Wilson lines. The number of the sites in the spin chain N corresponds to the number of the Wilson lines. The gauge group in the CS theory is the symmetry group of the magnet. Here it is the $\mathrm{SL}(2,\mathbb{R})$ and group manifold is an AdS_3 space. Hence the spin chain describes the motion of N points in AdS_3 space.

7. On the dual bulk representation of the reggeon

Let us make some comments about a dual gravity description of multi-colour QCD in the Regge limit. We can not present the complete solution of the problem due to lack of the proper (super)gravity background for nonsupersymmetric theories but mention some features which seem to be important for the behaviour of the scattering amplitudes. First of all we want to emphasize that in spite of the large-N_c limit one has to deal with Riemann surfaces whose genus is determined by the number of reggeons. In the scattering process the relevant geometry is captured by the membrane wrapped around the Riemann surface in the background Taub-NUT metric. Two transverse coordinates and two momenta are involved in this Taub-NUT background. It is this part of the total metric which provides the intercepts of the multi-reggeon states. We expect that back reaction of “scattering” membrane on the “vacuum” M5-brane can be neglected.

This dual gravity description should lead to the CFT description of reggeized gluons and their compound states on the transverse plane [51] which is not completely understood

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yet. It is tempting to make a conjecture that AdS_3 and AdS_2 spaces are the relevant geometries in the dual description. Indeed according the AdS/CFT correspondence the isometry group of the AdS manifold should coincide with the conformal group in the boundary theory. In the light-cone kinematics only the part of the full $\mathrm{SO}(4,2)$ conformal group survives which is $\mathrm{SL}(2,\mathbb{R})$ acting on the light cone. This is the symmetry group of the AdS_2 space. In the Regge limit the situation is more tricky. The dynamical $\mathrm{SL}(2,\mathbb{C})$ symmetry is not a subgroup of a full $\mathrm{SO}(4,2)$ symmetry of a classical QCD lagrangian. It emerges only at a quantum level. The corresponding AdS factor in the dual geometry for the Regge case can be identified with AdS_3 which in the euclidean version has the isometry group $\mathrm{SL}(2,\mathbb{C})$ indeed.

Let us formulate here a conjecture that in an AdS/CFT description reggeized gluons are described by AdS_3 singletons [52]. This conjecture is based on the observation that the correlation function of two reggeized gluons are logarithmic in transverse space and they can be identified with zero dimension logarithmic operators. It was shown some time ago that in the AdS/CFT correspondence the logarithmic operators on the boundary correspond to the singletons in the bulk [53].

To see the logarithmic correlator to appear let us consider the amplitude for the scattering of two quarks with infinite energy

$$W_{ij}^{i'j'} \equiv \langle 0|\mathcal{T} W_{+}^{i'i}(0) W_{-}^{j'j}(z)|0\rangle\,. \tag{7.1}$$

Here, Wilson lines W_+ and W_- are evaluated along infinite lines in the direction of the quark velocities v_1 and v_2, respectively:

$$\begin{aligned} W_+(0) &= P\exp\left(i\int_{-\infty}^{\infty} d\alpha\, v_1\cdot A(v_1\alpha)\right),\\ W_-(z) &= P\exp\left(i\int_{-\infty}^{\infty} d\beta\, v_2\cdot A(v_2\beta+z)\right), \end{aligned} \tag{7.2}$$

and the integration paths are separated by the impact vector $z=(0^+,0^-,\vec{z})$ in the transverse direction.

To understand the $z-$dependence, let us consider as an example the one-loop calculation of W in the Feynman gauge. One gets

$$\begin{aligned} W_{\text{1-loop}} &= I\otimes I + (t^a\otimes t^a)\frac{g^2}{4\pi^{D/2}}\Gamma\Big(\frac{D}{2}-1\Big)\lambda^{4-D}\times\\ &\times\int_{-\infty}^{\infty} d\alpha\int_{-\infty}^{\infty} d\beta\frac{(v_1v_2)}{[-(v_1\alpha-v_2\beta)^2+\vec{z}^{\,2}+i0]^{D/2-1}}\,. \end{aligned} \tag{7.3}$$

In this expression, gluon is attached to both Wilson lines at points $v_1\alpha$ and $v_2\beta+z$ and we integrate the gluon propagator $v_1^\mu v_2^\nu D_{\mu\nu}(v_1\alpha-v_2\beta-z)$ over positions of these points. To regularize IR divergences we introduced the dimensional regularization with $D=4+2\varepsilon$, $(\varepsilon>0)$ and λ being the IR renormalization parameter. There is another contribution to $W_{\text{1-loop}}$ corresponding to the case when gluon is attached by both ends to one of the Wilson lines. A careful treatment shows that this contribution vanishes. Performing integration over parameters α and β we get

$$W_{1\text{-loop}}(\gamma,\lambda^2\vec{z}^{\,2}) = I\otimes I + (t^a\otimes t^a)\frac{\alpha_{\rm s}}{\pi}(-i\pi\coth\gamma)\Gamma(\varepsilon)(\pi\lambda^2\vec{z}^{\,2})^{-\varepsilon}\,, \tag{7.4}$$

where $(v_1 v_2) = \cosh\gamma$.

The integral over α and β in (7.3) has an infrared divergence coming from large α and β. In the dimensional regularization, this divergence appears in $W_{1\text{-loop}}$ as a pole in $(D-4)$ with the renormalization parameter λ having a sense of an IR cutoff. We see that in an octet channel we got a logariphmic correlation which depends only on two transverse coordinates

$$(t^a\otimes t^a)\frac{\alpha_{\rm s}}{\pi}(i\pi\coth\gamma)\ln(\pi\lambda^2\vec{z}^{\,2}) \tag{7.5}$$

which allows us to make a conjecture that reggeized gluon is described by some logarithmic operator with zero dimension which correspond to a singleton field in AdS_3.

The arguments above suggest that the CFT at the transverse plane could be logarithmic [54]. Also it is tempting to relate the $\mathrm{SL}(2,\mathbb{C})$ symmetry of the AdS_3 space with the $\mathrm{SL}(2,\mathbb{C})$ symmetry of the original XXX spin chain but at this stage we do not have reasonable arguments in favor of this connection. Certainly these important issues deserve further investigation.

8. Conclusions

Let us summarize the obtained results. In this paper we proposed the stringy picture for multi-colour QCD in the Regge limit and on the light-cone in which hidden integrability plays the central role. Our approach is similar in many respects to the one which is known to be very successful in the description of the low energy limit of the $\mathcal{N}=2$ SUSY YM theories. In the Regge limit we encounter a new situation in comparison with a SUSY case. One has to develop the quantum picture involving the dynamics on a Riemann surface as the quasiclassical limit. We argued that the whole configuration relevant in the Regge limit follows from membrane wrapped around the spectral curve of the integrable system. Within our approach the Regge limit turns out to be in the same universality class as $\mathcal{N}=2$ superconformal SUSY YM with $N_f = 2N_c$ at the strong coupling regime. In both cases there is no natural place for the Λ_{QCD} type parameter: in SUSY case theory is conformal while in the Regge case we are dealing only with perturbative regime in generalized leading logarithmic approximation.

The WKB quantization conditions providing the compound multi-reggion spectrum and the spectrum of anomalous dimensions for composite light-cone operators are formulated in the stringy terms. They imply that energy of strings properly wrapped around the spectral curve has to be quantized. Note that since the quantization of the integrable system amounts to the quantization of the moduli space of the complex structures of the Riemann surface quantum gravity in two dimensions is involved. The correspondence operator $\Leftrightarrow$ Riemann surface plays an important role in our approach and is some new step toward a complete formulation of the gauge/string correspondence.

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One more issue which certainly deserves further investigation is the meaning of the unitary Froissart boundary. Let us emphasize that to get the scattering amplitude one has to sum over all Riemann surfaces of arbitrary genus, corresponding to the different number of reggeons N

$$A(s,t) = \sum_{N\geq 2} A_N(s,t) = \sum_{\text{genus=N}-2} A_g(s,t) \tag{8.1}$$

in perfect agreement with the string picture. Hence, if the picture suggested for the Froissart saturation in [26] is correct it would mean that the process of the black hole creation in the s-channel is equivalent to the closed string propagation in nontrivial background in the t-channel.

Let us mention a few other important questions which remain open. One of them is to find more clear physical identification of the M-theory 10-th dimension involving the geometry of the spectral curve. At a moment we can accurately introduce it only via the Bloch-Floquet factor of the Baker-Akhiezer function but more physical interpretation is highly desirable. The related question concerns the interpretation of the "magnetic" quantum numbers amounted from the spectral curves. It would help in a wider application of the dualities in the Regge limit and possibly could shed the additional light on the $s-t$ duality in the high energy amplitudes.

Acknowledgments

We are grateful to S. Derkachov, Yu. Dokshitzer, A. Kaidalov, A. Leonidov, A. Manashov, N. Nekrasov and F. Smirnov for the useful discussions. The work of A.G. is partially supported by grants CRDF-01-2108 and INTAS-00-334, I.K. is supported in part by PPARC rolling grant PPA/G/O/1998/00567 and EC TMR grant HPRN-CT-1999-00161, both I.K. and G.K. are partially supported by a joint CNRS-Royal Society grant.

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